AN INVESTIGATION OF THE SIDE THRUST AVAILABLE FROM PROPELLER SHROUDING SURFACES

by
SAMUEL JAMES GORDON
B. S. E., University of Michigan
(1959)

SUBMITTED IN PARTIAL FULFILLMENT OF THE
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Signature of Author . . . .
Department of Naval Architecture and
Marine Engineering, May 20, 1966

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Thesis Supervisors

Accepted by . . . .
Chairman, Departmental Committee on Graduate Studies

ABSTRACT

AN INVESTIGATION OF THE SIDE THRUST AVAILABLE FROM PROPELLER SHROUDING SURFACES

by

SAMUEL JAMES GORDON

Submitted to the Department of Mechanical Engineering and to the Department of Naval Architecture and Marine Engineering in partial fulfillment of the requirements for the degree of Master of Science in Mechanical Engineering and the Professional Degree of Naval Engineer.

A theory and analytical method are developed for the determination of the circumferential circulation distribution, over an annular airfoil of arbitrary included angle. The resultant circulation distribution is then used to determine the forces acting on the airfoil surface.

The theory is based on the "L-Method" developed by J. Weissinger, which consists of replacing the lifting surface by a single concentrated ring vortex and an associated system of trailing vortices. The method presented here is compared to other theories in the limiting case of the complete symmetrical annulus. It is shown to give the same result as Weissinger found using the "L-Method." This, in turn, is shown to be a very close approximation to the more accurate theories and to experimental results.

A sample calculation is made, based on the method presented, of the forces acting on a semicircular propeller shroud, to determine its usefulness as a steering device in lieu of the conventional ship's rudder. The results are compared with those for a fully movable spade type rudder with area equal to that of the shroud and with a high lift to drag ratio. It is found that the predicted side thrust from the shroud is more than twice the maximum predicted for the rudder. Additionally the shroud is found to give a thrust augmentation equal to approximately 10% of the propeller thrust while the rudder absorbs a minimum of 1% of the propeller thrust and a maximum of 30%.

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CONTENTS

Introduction:

The literature contains a good deal of material on the theory of annular airfoils and ducted propellers. Authors such as Morgan[2] and Weissinger[1] have treated the problem of a symmetrical ring airfoil at an angle attack, alone, as well as in the presence of other objects such as propellers and hubs. Weissinger has also treated the case of a complete annular airfoil with deflected control surfaces. There seems to be no evidence of any previous work on incomplete rings.

Since the incomplete shroud seemed, from an intuitive standpoint, to provide a very effective means of steering a ship, I constructed a small self-propelled model and fitted it with a semi-circular propeller shroud. The model is illustrated in Figure 1. By rotating the shroud about the propeller axis the side thrust developed by the shroud could be directed so as to cause the model to turn to the right or left. The steering device worked very well on the model and provided the stimulus for this investigation.

The theory developed here, I feel, provides a means for making a reasonably accurate prediction of the performance of either complete or incomplete shrouds with a circumferential variation in circulation distribution.

Conclusions:

The results of this sample calculation indicate that partial shrouds or complete shrouds with controllable circulation and lift distributions can be used to good advantage as steering devices for fluid born vehicles such as ships and ground effect machines.

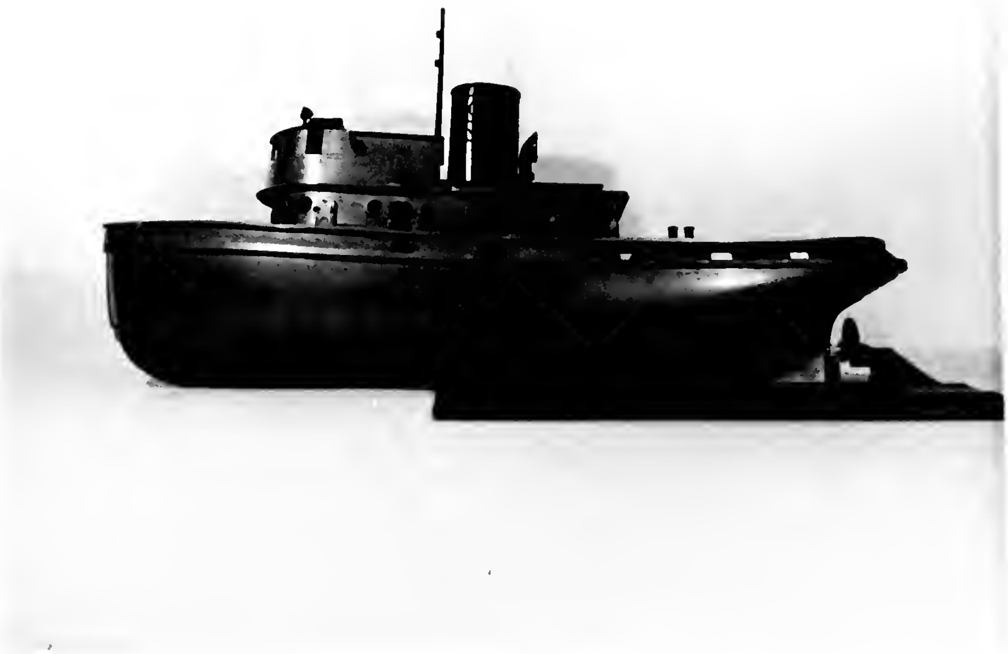
The following improvements in performance over that obtained with conventional rudders are predicted:

1. Substantial elimination of rudder drag at all turning rates.
2. Thrust augmentation from the shroud.
3. Improved steering capability with little or no way on.
4. Significant decrease in turning diameter with an equivalent size turning device.

The following possible shortcomings in the application of shrouds as steering devices require consideration and investigation:

1. Decrease in propeller loading necessitated by shroud induced cavitation.
2. Propeller induced shroud vibration.
3. Possibility of loss of steering capability with engines stopped.

It is hoped that further theoretical work in this area along with experimental testing will be carried out and that the apparent benefits to be gained from the use of propeller shrouds for steering will be employed in practice.



SIDE VIEW OF MODEL

VIEW SHOWING SHROUD IN POSITION FOR TURNING TO THE RIGHT

FIGURE I

Procedure: The method utilized for determining the forces on the shroud is that developed by J. Weissinger and known as the "L-Method" or "Weissinger's Method." The following basic assumptions are made:

(a) The fluid is incompressible and baratropic.

(b) Body forces, such as gravity, are neglected.

(c) The flow is steady.

(d) Viscous effects are confined to a negligibly thin boundary layer which does not separate on the shroud.

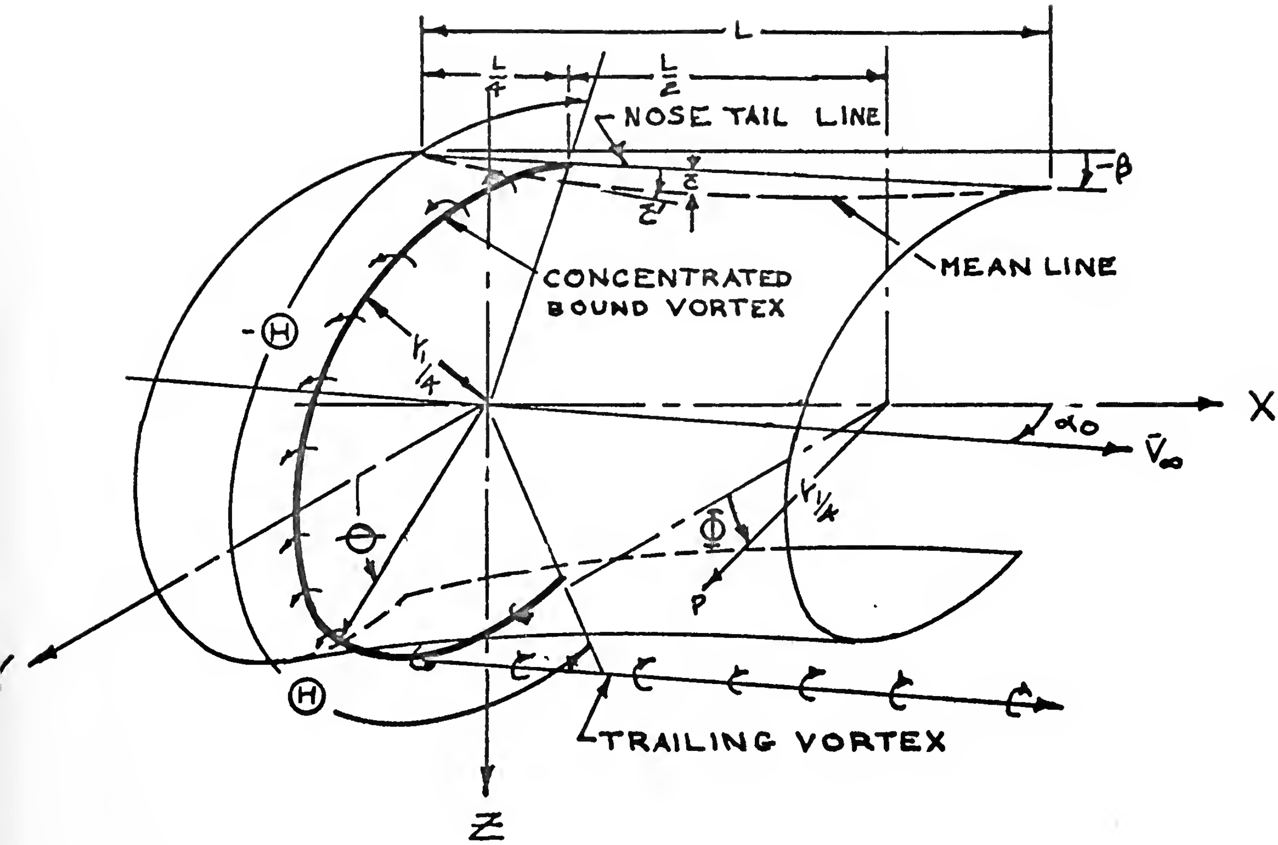
The method consists of the replacement of the lifting surface by a single, concentrated bound vortex at the 1/4 chord point and an associated system of trailing vortices as shown in Fig. 2. The circulation distribution, along the bound vortex, is determined by requiring that the velocity normal to the mean line at the 3/4 chord point must be zero. Since the circulation distribution varies along the span of the shroud, a system of trailing vortices will be shed off which will form a cylindrical sheet of radius equal to that of the concentrated vortex ring. The boundary condition is linearized by requiring that the quotient of the radial velocity and the free stream velocity, at the 3/4 chord point, along this cylindrical sheet, be equal to the slope of the section at the 3/4 point. That is:

P1: $$\left.\frac{\vec{q}_{r_s} + \vec{q}_{r_p}}{\vec{V}_{xp}}\right|_{(1,\ \Phi,\ \lambda_{3/4})} = \vec{r}\left[\vec{c}_1'(\Phi, \lambda_{3/4}) - \alpha(\Phi)\right]$$

Once the circulation distribution is known, the radial force and axial forces per unit length at any section can be found by the "Kutta-Joukowski Law" which, in this case, states:

P2: $$\vec{F}_r(\theta) = -\rho \vec{V}_{xp}(1, \theta, 0)\, \Gamma_{(\theta)}$$

P3: $$\vec{F}_x(\theta) = \rho\left[\vec{q}_{r_p}(1, \theta, 0) + \vec{q}_{r_s}(1, \theta, \lambda_{1/4})\right]\Gamma_{(\theta)}$$



(Thickness not shown)

FIGURE 2

L	=	Shroud length.
Θ	=	Limiting half angle of shroud.
θ	=	Angular coordinate along concentrated vortex in 1/4 chord plane.
$r_{1/4}$	=	Radius of concentrated vortex in 1/4 chord plane.
P	=	Pivotal point at radius equal to $r_{1/4}$ in 3/4 chord plane.
Φ	=	Angular coordinate of pivotal point in 3/4 chord plane.
$\bar{c}$	=	Displacement of mean line relative to nose tail line, nondimensionalized on L.
$\bar{c}'$	=	Slope of mean line relative to nose tail line.
λ	=	$x/r_{1/4}$ = Nondimensionalized axial coordinate
ρ	=	$r/r_{1/4}$ = Nondimensionalized radial coordinate
$-\beta$	=	Convergence angles of nose-tail cone
α_o	=	Angle of attack in plane of symmetry.

For a shroud which is symmetrical about $\theta = 0$, the resultant radial force $\vec{R}$ lies in the plane of symmetry and is equal to the total lift acting on the shroud. $\vec{R}$ can be found by integrating the components of $\vec{F}_r(\theta)$, parallel to the plane of symmetry, over the span as follows:

P4: $$T(0) = V_{1/4} \int_{-\Theta}^{\Theta} \vec{F}_r(\theta) \cos \theta \, d\theta$$

Similarly, the total induced drag $\vec{D}$, acting on the shroud, is found as follows:

P5: $$\vec{D}_i = V_{1/4} \int_{-\Theta}^{\Theta} F_x(\theta) \, d\theta$$

The problem reduces to the determination of the circulation distribution for a given propeller geometry and loading and a chosen shroud geometry.

The circulation distribution is represented as a Fourier cosine series.

A-23: $$\Gamma_{(\theta)} = \sum_{n=0}^{\infty} A_n \cos (n\theta)$$

It is to be noted here that θ is the actual polar angle measured from the plane of symmetry. For an incomplete shroud the condition that the circulation vanish at the tips is satisfied by requiring that the sum of all terms in the series vanish rather than that each term vanish individually. The resultant radial velocity induced by the shroud, at a point on the trailing vortex sheet, is determined as a series equation in terms of a set of coefficients and integrals.

A-26: $$\vec{q}_{r_s} = \vec{r} \sum_{n=0}^{\infty} \sum_{m=1}^{6} C_{nm} I_{nm}$$

where C_{nm} are coefficients involving the parameters A_n and I_{nm} are geometric integrals.

Since four terms in the series representation of $\Gamma_{(\theta)}$ (Eq. A-23) will provide a representation of more than sufficient accuracy for our purposes, we will restrict our consideration to n = 0, ..., 3. We must, therefore, evaluate a maximum of 24 coefficients, C_{mn}, and 24 integrals, I_{mn} for each angular co-ordinate (Φ) at which we desire to determine $\vec{q}_{r_s}$. These can then be combined, using Eq. A-26, to form an equation in terms of the coefficients, A_n, of Eq. A-23. If we let $n = 0, \ldots, n_{max}$, $n_{max} \leq 3$, we must evaluate $\vec{q}_{r_s}$ at $(n_{max} - 1)$ values of Φ by using Eq. A-26.

Knowing the propeller velocity field for a given propeller geometry and loading, and knowing the shroud geometry we can evaluate $\vec{q}_{r_s}$ at the same $(n_{max} - 1)$ values of Φ by Eq. P-1. In the case of an incomplete shroud we have an additional boundary condition to satisfy and that is that $\Gamma_{(\pm\Theta)} = 0$. This gives us an additional equation resulting in a set of n_{max} simultaneous equations in the n_{max} unknowns $A_0, \ldots, A_{n\,max}$. For a complete shroud we must evaluate $\vec{q}_{r_s}$ at an additional point to give us n_{max} equations.

The solution to this set of equations provides the desired representation of the circulation distribution and permits the determination of the side thrust available from the propeller shroud as well as the drag induced by it.

Weissinger has shown, in Ref. 1, and it is shown here in Appendix C, that this method yields the exact value of total lift for a complete ring airfoil in the limit of very high or very low aspect ratios ($L/2r_{1/4}$) and yields a remarkably good approximation for arbitrary aspect ratios falling between these limits.

It is to be noted that this method gives the correct solution only for airfoils of zero thickness as it takes account only of the circulation and not of the thickness. The effects of thickness must be accounted for separately; however, their

effect on the total force acting on the surface is small in comparison to that of circulation as shown in Fig. 22, Ref. 2. (Reproduced here as Fig. A-5.)

Sample Calculations:

Consider a propeller with the following characteristics:

$D_P = 12.0'$ $\quad$ $SHP = 6000$

$(1 - t) = 0.80$ $\quad$ $(1 - W) = 0.70$

$V_S = 20$ KTS $\quad$ $V_a = V_\infty = 14$ KTS; $V_\infty = 23.65$ F.P.S.

$n_p = 0.625$ $\quad$ $e_{rr} = 1.00$

$N = 200$ RPM

$$THP = SHP \quad n_p e_{rr} = 3750 \text{ HP} = \frac{TVa}{550}$$

$$C_T = \text{Thrust Coeff.} = \frac{T}{\pi R_P^2 \frac{1}{2} \rho V_a^2}$$

$$C_T = \frac{2(THP) \times 550}{\pi R_P^2 \rho (V_\infty)^3} \quad ; \quad \rho = 1.99 \text{ slugs/ft}^3$$

$$C_T = \frac{2(3750) \times 550}{\pi(6.0)^2 (1.99)(23.65)^3} = \frac{7.5 \times 5.5 \times 10^6}{\pi(3.6)(1.99)(2.365)^3 \times 10^4}$$

$$C_T = 1.385$$

Assume the circulation distribution over the propeller is constant; radial and axial velocity fields for this case are tabulated in Tables 3 and 4, Ref. (8).

Shroud Geometry:

Consider a shroud of constant chord, camber and convergence angle, and assume, for the purposes of this calculation, that the velocities induced by the propeller, at the shroud vortex sheet, are equivalent to those induced on a cylinder of radius R_P. Take:

$$\frac{(L/2)}{R_P} = 0.8 \ ; \ L = 9.6'$$

Choosing NACA a = 0.6 meanline as tabulated in Ref. (7)

$$\alpha_i = 2.58° \ ; \quad \vec{c}\,' \,\Big|\, x = \frac{L}{2} = +0.1827$$

$$\frac{\vec{c}}{L} \,\Big|\, x = \frac{L}{4} = 0.0737$$

Referring to Table 3, Ref. (8) at $\frac{r}{R} = 1.0$; $\frac{x}{R} = -0.4$ (shroud leading edge)

$$\frac{\vec{q}_{xp}}{\vec{v}_\infty C_T} = +0.08 \ ; \ \vec{q}_{xp} = 0.08\,(1.385)\,\vec{v}_\infty$$

$$\vec{q}_{xp} + \vec{v}_\infty = 1.11\,\vec{v}_\infty$$

Referring to Table 4, Ref. (8):

$$\frac{\vec{q}_{rp}}{\vec{v}_\infty C_T} = -\,0.086 \ ; \ \vec{q}_{rp} = -0.086\,(1.385)\,\vec{v}_\infty$$

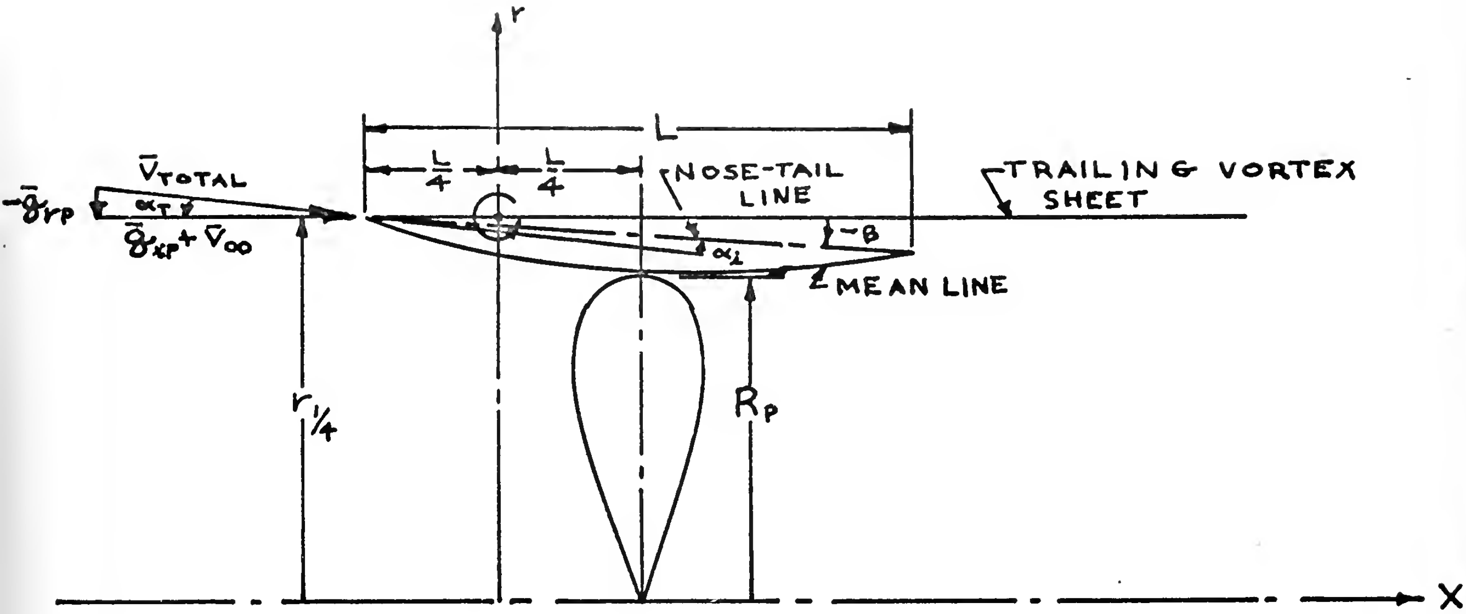


FIGURE 3

For ideal angle of attack:

$$\alpha_T = -\beta + \alpha_i = \tan^{-1}\left(\frac{q_{rp}}{q_{xp} + v_\infty}\right) = \tan^{-1}\left(\frac{0.119}{1.11}\right)$$

$$\alpha_T = -\beta + \alpha_i = 6.12°$$

$$-\beta = 6.12° - 2.58° = 3.54° = 0.620 \text{ Radians}$$

The shroud will have the following characteristics:

$$r_{1/4} = 6.0' + (0.737)(9.6) + 0.62\,(9.6/4) = 6.857'$$

$L = 9.60'$

$\Theta = 90°$

Mean line: N. A. C. A. a = 0.6

$-\beta = 3.54°$

$$\lambda_{3/4} = \frac{L/2}{r_{1/4}} = \frac{4.80}{6.857} = 0.70$$

Boundary Conditions for Determining Thrust:

(1) at $\theta = \pm \Theta; \Gamma_{(\theta)} = 0$

(2) at $\lambda_{3/4}$; $\dfrac{\vec{q}_{rs} + \vec{q}_{rp}}{\vec{q}_{xp} + \vec{v}_{\infty}} = \vec{c}^{\,1} - (-\beta) = 0.1827 - 0.062$

$$q_{rs} = (q_{xp} + v_{\infty})(0.1207) - q_{rp}$$

From Table 3, Ref. (8):

$$\frac{q_{xp}}{v_{\infty} C_T} = 0.155$$

$$q_{xp} = 0.155\ (1.385)\ v_{\infty} = 0.215\ v_{\infty}$$

From Table 4, Ref. (8):

$$\frac{q_{rp}}{v_{\infty} C_T} = -0.136$$

$$q_{rp} = -0.136\ (1.385)\ v_{\infty}$$

$$q_{rs} = (0.147 + 0.188)\ v_{\infty} = 0.335\ v_{\infty}$$

$$q_{rs} = 0.335\ (23.65) = 7.93 \text{ F.P.S.}$$

Take:

$\Phi_1 = 0$

$\Phi_2 = \pi/4$

$\Phi_3 = \pi/3$

The set of simultaneous equations determining the four coefficients in the cosine series representation of the circulation distribution are:

$$0.025\ A_0 + 0.0741\ A_1 + 0.0963\ A_2 + 0.1739\ A_3 = 7.93$$

$$0.0235\ A_0 + 0.0422\ A_1 + 0.0198\ A_2 - 0.0450\ A_3 = 7.93$$

$$0.0214\ A_0 + 0.0248\ A_1 + 0.0165\ A_2 - 0.153\ A_3 = 7.93$$

$$1.0\ A_0 \quad + 0\ A_1 \quad - 1.0\ A_2 \quad + 0\ A_3 \quad = 0$$

Solving simultaneously:

$$A_0 = A_2 = 338.0$$

$$A_1 = -249.2$$

$$A_3 = -84.6$$

$$\Gamma_{(\theta)} = 338 - 249.2 \cos(\theta) + 338 \cos(2\theta) - 84.6 \cos(3\theta)$$

By the Kutta-Joukowski Law

$$\vec{L}_{(\alpha T)} = -\rho \frac{\left(V_\infty + q_{xp}\big|_{x=o}\right)}{\cos \alpha_T} \, r_{1/4} \int_{-\pi/2}^{\pi/2} \Gamma_\theta \cos\theta \, d\theta$$

By Ref. (8):

$$\left.\frac{q_{xp}}{V_\infty C_T}\right|_{x=o} = 0.10 \ ; \ q_{xp} = 0.1385\ V_\infty$$

$$\vec{L}_{(\alpha T)} = \underline{\underline{-188,500}} \text{ Lbs.}$$

$\vec{L}_{(\alpha T)}$ is positive when directed radially outward, perpendicular to V_{total}

Determination of the Induced Drag:

$D_i(\alpha T) = - T(0) \tan \alpha_i$

$D_i(\alpha T) = + 188,500 (0.045) = 8,480$ Lbs.

The induced drag is in a direction parallel to V_{total} .

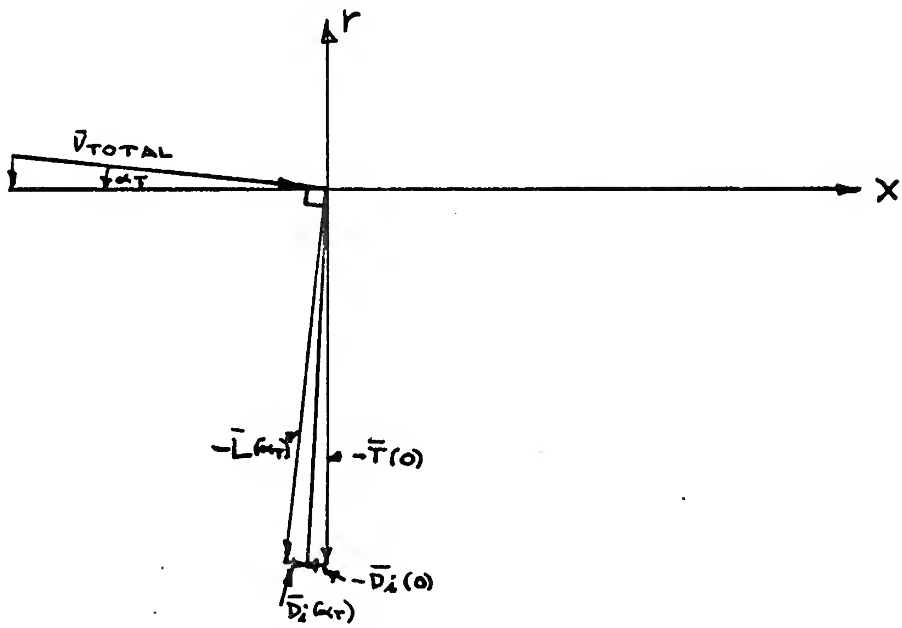


FIGURE 4

$-T(0) = -L_{(\alpha T)} \cos \alpha_T + D_i (\alpha T) \sin \alpha T$

$-T(0) = 188,500 (0.9943) + 8480 (0.1067)$

$-T(0) = 187,500 + 940 = 188,440$

$\therefore -T_{(0)} \approx -L_{(\alpha T)} = \underline{\underline{188,500}}$ Lbs.

$-D_i(0) = -L_{(\alpha T)} \sin (\alpha T) - D_i(\alpha T) \cos \alpha T$

$-D_i(0) = 20,100 - 8,400 = \underline{\underline{11,700}}$ Lbs.

Form Drag:

For an N. A. C. A. 66-210 section

By Ref. (7):

$$C_D \text{ @ } \alpha = 0 = 0.0094$$

where

$$C_D\Big|_{\alpha=o} = \frac{D_f}{\frac{1}{2}\,\rho(V_\infty + q_{xp})^2\,\pi\, r_{1/4}\, L}$$

$$\therefore\ D_f = 0.0094\ (1.0)\ [(1.215)\ (23.65)]^2\ \pi\ (6.857)\ (9.6)$$

$$D_f = 9.4\ (0.826)\ \pi\ (6.857)\ (9.6)$$

$$D_f = 1600 \text{ Lbs.}$$

Taking D_f as acting in the +x direction

$$D_{total} = D_i + D_f = -11,700 + 1600$$

$$D_{total} = -9,100 \text{ Lbs.}$$

Final Predicted Thrust from this Shroud:

Ahead thrust = 9,000 Lbs.

Side thrust = 188,000 Lbs.

Comparison with a Conventional Rudder:

Assuming that $\alpha/\delta = 5/7$ where:

α = angle of attack

δ = rudder angle.

For a rudder of Aspect Ratio = 3, for the best L/D ratio and of area equal to the area of the shroud:

A = area = $(6.86 + 0.48)\ \pi\ (9.6) = 221\ \text{Ft.}^2$

AR = Aspect Ratio = $(2S)^2/A = 3$

S = Span

$$S^2 = \frac{3}{4}(221) = 166 \ ; \ S = 12.88'$$

$$C_L = \frac{L}{(\rho/2)\, V_\infty^2\, A} \qquad C_D = \frac{D}{(\rho/2)\, V_\infty^2\, A}$$

By Figure 62, Ref. (9):

δ	α	$\delta-\alpha$	C_L	L	C_D	D
0	0	0	0	0	0.01	930
22.4°	16°	6.4°	0.830	77,000	0.096	8,900
28.0°	20°	8.0°	0.980	91,000	0.163	15,120

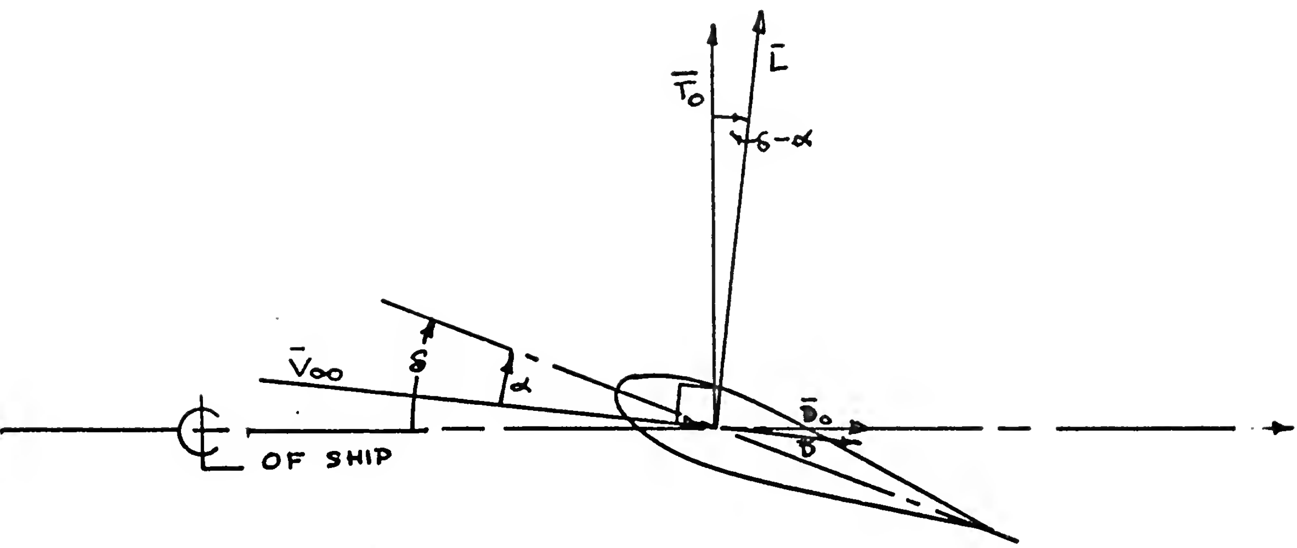


FIGURE 5

T_o = Side Thrust = $L \cos(\delta - \alpha) - D \sin(\delta - \alpha)$

D_o = Drag = $L \sin(\delta - \alpha) + D \cos(\delta - \alpha)$

$\delta = 22.4°$:

$T_o = 76,000$

$D_o = 17,400$

$\delta = 28°$: (Breakdown)

$T_o = 88,000$

$D_o = 27,500$

$\delta = 0$:

$T_o = 0$

$D_o = 930$

SYMBOLS:

A_n	=	Coefficient in series representation of circulation distribution
$\vec{C}$	=	Nondimensionalized height of mean line above the nose tail line.
$\vec{C}'$	=	Slope of the mean line relative to the nose tail line
C_D	=	Drag coefficient $= \dfrac{D}{\pi R_P^2 \frac{1}{2} \rho V^2}$
C_L	=	Lift coefficient $= \dfrac{T(0)}{\rho V_\infty^2 \, r_{1/4} \lambda}$
C_{nm}	=	Coefficient in series representation of induced radial velocity.
C_T	=	Thrust coefficient $= \dfrac{T}{\pi R_P^2 \frac{1}{2} \rho V_a^2}$
$\vec{D}$	=	Position vector from vortex element to pivitol point.
$\vec{D}_f$	=	Form Drag
$\vec{D}_i(\alpha)$	=	Induced Drag
$E(\varphi_p, k)$	=	Incomplete elliptic integral of the first kind
$\vec{F}_r$	=	Radial force per unit arc length
$\vec{F}_x$	=	Axial force per unit arc length
$\vec{i}, \vec{j}, \vec{k}$	=	Unit vectors in Cartesian coordinates
I_{nm}	=	Integral occurring in the series representation of induced radial velocity.
k^2	=	Modulus in elliptic functions
$K(k)$	=	Complete elliptic integral of the first kind
ℓ	=	$\dfrac{\lambda^2 + 2}{2}$ = constant for a particular pivotal point
L	=	Chord length
$\vec{L}(\alpha)$	=	Lift force
N	=	Propeller R. P. M.

P	=	An arbitrary field point where the velocity is to be determined.
q	=	Induced velocity.
$\vec{q}_b$	=	Velocity induced by the bound vortex distribution.
$\vec{q}_{rp}$	=	Radial velocity induced by the propeller.
$\vec{q}_{rs}$	=	Radial velocity induced by the shroud.
$\vec{q}_T$	=	Velocity induced by the trailing vortex distribution.
$\vec{q}_{xp}$	=	Axial velocity induced by the propeller.
$\vec{q}_{xs}$	=	Axial velocity induced by the shroud.
r	=	Radial coordinate.
$\vec{r}, \vec{\varphi}, \vec{k}$	=	Unit vectors in polar coordinates.
$\vec{R}$	=	Resultant force acting on the shroud.
SHP	=	Shaft horsepower.
t	=	Dummy variable = cos ν.
t	=	Thrust deduction.
THP	=	Thrust horsepower.
$\vec{T}_{(\theta)}$	=	Side thrust.
$\vec{v}_a$	=	Speed of advance through medium.
$\vec{V}_\infty$	=	Free stream velocity.
v_1, v_2, v_3	=	Functions used in evaluating elliptic integrals.
$\vec{V}_{\alpha p}$	=	$\vec{V}_\infty + \vec{q}_{xp}$ = axial velocity of propeller stream.
w	=	Wake fraction.
x, ξ	=	Axial coordinate.
α_i	=	Induced angle of attack at the 1/4 chord point.
α_o	=	Angle of attack measured from incoming stream to shroud axis in plane of symmetry.
β	=	Convergence angle of mean line.
γ	=	Circulation per unit length (positive in the positive coordinate direction.)

Γ = Circulation (positive in the positive coordinate direction).

$\epsilon = \dfrac{\xi}{r_{1/4}}$ = Nondimensionalized axial coordinate.

E_{rr} = Relative rotative efficiency.

η_p = Propeller efficiency.

θ, Φ = Polar angle.

Θ = Limiting half angle of shroud.

λ = $x/r_{1/4}$ = Nondimensionalized axial coordinate.

ν = Angular variable of integration = $\pi - (\Phi - \theta)$.

ξ = Axial coordinate.

$\Pi(\varphi_p, \alpha^2, k)$ = Incomplete elliptic integral of the third kind.

ρ = Density of fluid.

$\rho = r/r_{1/4}$ = Nondimensionalized radial coordinate

φ_p = Amplitude of angular argument in elliptic functions.

Φ = Angular coordinate of the pivotal point.

Subscripts:

B	Due to bound vortex
n, m, p	Indices
p	Due to propeller
r	In radial direction
s	Due to shroud
T	Due to trailing vortices
x	In axial direction
θ, φ	In tangential direction
1/4	At 1/4 chord point
3/4	At 3/4 chord point

REFERENCES

1. Weissinger, J. "Zur Aerodynamik des Ringflügels, I. Die Druckverteilung dunner, fast drehsymmetrischer Flügel in Unterschallströmung," Deutsche Versuchsanstalt für Luftfahrt, E. V. Bericht Nr. 2, Mulheim (Sep 1955).
2. Morgan, W. B., "Prediction of the Aerodynamic Characteristics of Annular Airfoils," David Taylor Model Basin Report 1830, January 1965.
3. Chemical Rubber Co., "Standard Mathematical Tables."
4. Ryshik, I. M., Gradstein, I. S., "Tables of Series, Products and Integrals," Veb Deutscher Verlag der Wissenschaften, 1963.
5. Byrd, Paul F., Friedman, Morris, D., "Handbook of Elliptic Integrals for Engineers and Physicists," Springer Verlag, Berlin, 1954.
6. Morgan, W. B., "A Theory of the Ducted Propeller with a Finite Number of Blades," University of California, Institute of Engineering Research, Berkeley, May 1961.
7. Abbot, J. H., Von Doenhoff, A. E., "Theory of Wing Sections," Dover, 1959.
8. Hough, G. R., Ordway, D. E., "The Generalized Actuator Disk," TAR-TR 6401, Therm Advanced Research Inc., Ithaca, N. Y., January 1964.
9. Whicker, L. F., Fehlner, L. F., "Free Stream Characteristics of a Family of Low Aspect Ratio, All-Movable Control Surfaces for Application to Ship Design," David Taylor Model Basin Report 933, December 1958.

APPENDIX A

DEVELOPMENT OF THE INTEGRAL EQUATION FOR THE VELOCITY INDUCED BY THE SHROUD

I. <u>Shroud Vortex System:</u>

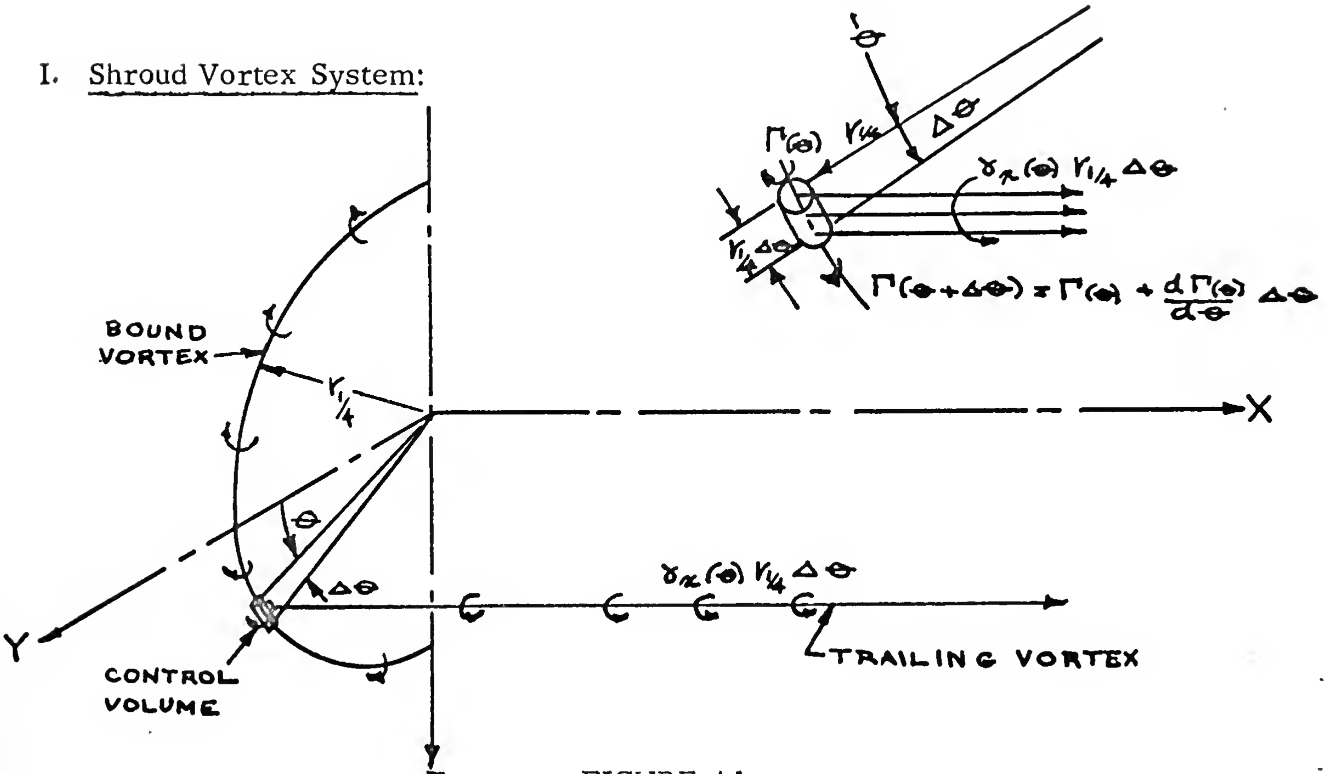


FIGURE A1

Consider the control volume enclosing the incremental arc of the bound vortex distribution lying between the polar angles θ and θ + dθ of Fig. A1. Helmholtz vortex theorem states that vortex filaments cannot end in the fluid. This requires that the circulation around the vortex filaments leaving the control volume must equal that of the vortex filaments entering. This theorem implies conservation of vorticity and, in terms of the circulation states that:

$$\Gamma(\theta + \Delta\theta) + \gamma_x(\theta)\, r_{1/4}\, \Delta\theta = \Gamma(\theta)$$

$$\Gamma(\theta) + \left(\frac{d\Gamma_\theta(\theta)}{d\theta}\right) \Delta\theta + \gamma_x(\theta)\, r_{1/4}\, \Delta\theta = \Gamma(\theta)$$

Therefore:

A-1:

$$\gamma_x(\theta) = -\frac{1}{r_{1/4}}\frac{d\Gamma(\theta)}{d\theta}$$

II. Velocity Induced at a Field Point by the Shroud Vortex System:

By the law of "Biot-Savart":

A-2:

$$d\vec{q} = \frac{1}{4\pi}\frac{\Gamma(ds \times \vec{D})}{|D|^3}$$

where: $d\vec{q}$ = incremental velocity induced at a particular field point or "pivotal point".

$\vec{ds}$ = directed incremental element of the vortex line.

Γ = magnitude of the circulation around the vortex element $\vec{ds}$.

$\vec{D}$ = vector displacement from the vortex element to the pivotal point.

Integrating A2, it follows that:

A-3:

$$\vec{q}_B(r, \Phi, X) = \frac{1}{4\pi}\int_{-\Theta}^{\Theta}\frac{\Gamma(\theta) r_{1/4}\, \vec{d\theta} \times \vec{D}}{|D|^3}$$

and

A-4:

$$q_T(r, \Phi, X) = \frac{1}{4\pi}\int_{-\Theta}^{\Theta}\int_{0}^{\infty}\frac{\gamma_x(\theta) r_{1/4}(\vec{d}_x \times \vec{D})\, d\theta}{|D|^3}$$

If the pivotal point coincides with a point on the vortex system, one or both of the integrals must be evaluated as the Cauchy principle value to exclude the singularity at the pivotal point.

III. Integral Equations for Velocity Induced by the Bound Vortex Distribution:

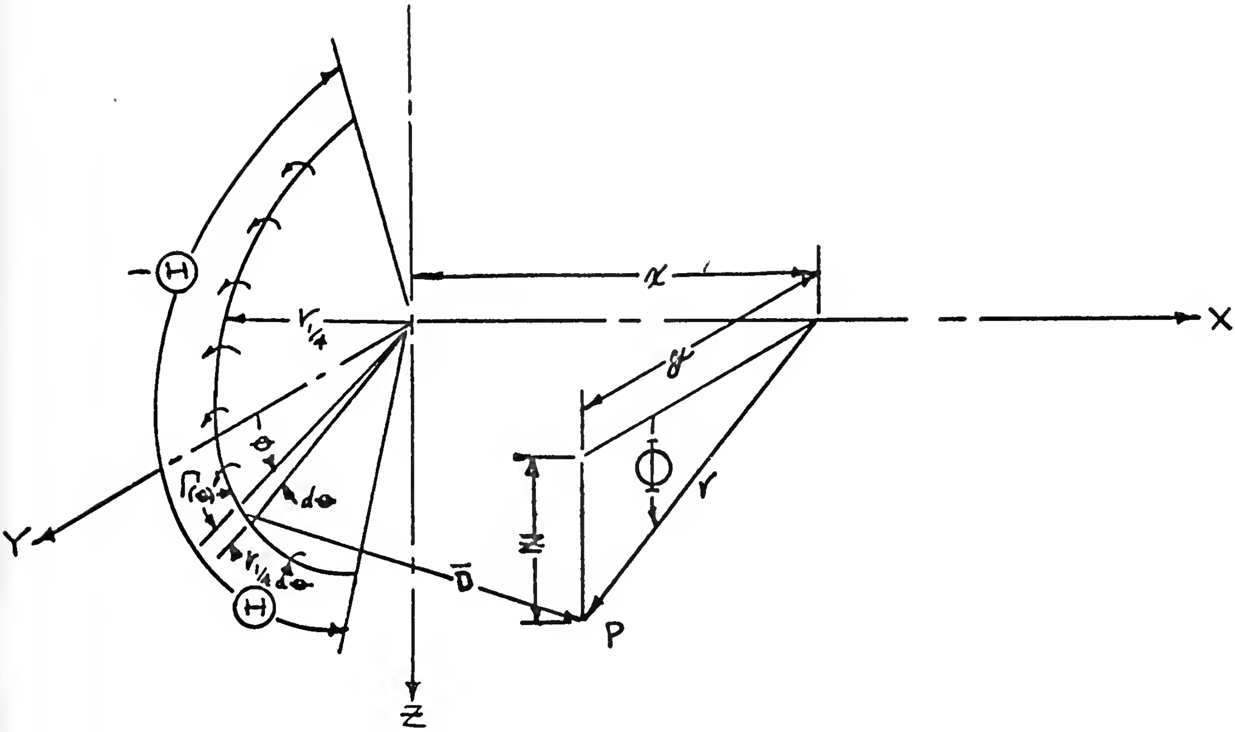


FIGURE A2

"P" is an arbitrary pivotal point where the induced velocity is to be determined. p has cartesian coordinates (x, y, z) and polar coordinates (r, Φ, x).

By the law of Biot-Savart:

A-4:

$$\vec{d}q_B = \frac{1}{4\pi}\,\Gamma\; r_{1/4}\;\frac{\vec{d}\theta \times \vec{D}}{|D|^3}$$

In cartesian coordinates:

A-5:

$$d\vec{\theta} = d\theta\,[0\vec{i} - \sin\theta\vec{j} + \cos\theta\,\vec{k}\,]$$

$$\vec{r} = r\,[0\vec{i} + \cos\Phi\vec{j} + \sin\Phi\,k]$$

A-6:

$$\vec{D} = (x)\vec{i} + [r\cos\Phi - r_{1/4}\cos\theta]\vec{j} + [r\sin\Phi - r_{1/4}\sin\theta]\vec{k}$$

$$d\vec{\theta} \times \vec{D} = -d\theta\left[\sin\theta\,(r\sin\Phi - r_{1/4}\sin\theta) + \cos\theta\,(r\cos\Phi - r_{1/4}\cos\theta)\right]\vec{i}$$

$$+ d\theta\,[x\cos\theta\,]\vec{j}$$

$$+ d\theta\,[x\sin\theta\,]\,\vec{k}$$

$$|D| = \left[x^2 + r^2 + r_{1/4}^2 - 2r\,r_{1/4}\cos(\Phi - \theta)\right]^{1/2}$$

Nondimensionalizing:

A-7:

$$\frac{r}{r_{1/4}} \equiv \rho\,;\ \frac{x}{r_{1/4}} \equiv \lambda$$

A-8:

$$|D| = r_{1/4}\,[\lambda^2 + \rho^2 + 1 - 2\rho\cos(\Phi - \theta)]^{1/2}$$

A-9:

$$d\vec{\theta} \times \vec{D} = r_{1/4}\left\{-[\rho\cos(\Phi - \theta) - 1]\,d\theta\,\vec{i} + \lambda\cos\theta\,d\theta\,\vec{j} + \lambda\sin\theta\,d\theta\,\vec{k}\right\}$$

Substituting in A-7, A-8, A-9 into A-4:

$$d\vec{q}_{bx} = -\frac{\Gamma(\theta)}{4\pi r_{1/4}}\ \frac{[\rho\cos(\Phi - \theta) - 1]\,d\theta}{[\lambda^2 + \rho^2 + 1 - 2\rho\cos(\Phi - \theta)]^{3/2}}\ \vec{i}$$

$$d\vec{q}_{by} = \frac{\Gamma(\theta)\,\lambda\cos\theta\,d\theta}{4\pi r_{1/4}\,[\lambda^2 + \rho^2 + 1 - 2\rho\cos(\Phi - \theta)]^{3/2}}\ \vec{j}$$

$$d\vec{q}_{bz} = \frac{\Gamma(\theta)\, \lambda \sin\theta \, d\theta}{4\pi r_{1/4} \left[\lambda^2 + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)\right]^{3/2}} \vec{k}$$

$$dq_{br} = d\vec{q}_b \bullet \frac{\vec{r}}{|r|}$$

$$dq_{br}\vec{r} = \left[\frac{\Gamma(\theta)\, \lambda \cos\theta \cos\Phi \, d\theta}{4\pi r_{1/4} \left[\lambda^2 + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)\right]^{3/2}} \right.$$

$$\left. + \frac{\Gamma(\theta)\, \lambda \sin\theta \sin\Phi \, d\theta}{4\pi r_{1/4} \left[\lambda^2 + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)\right]^{3/2}} \right] \vec{r}$$

$$d\vec{q}_{br} = \frac{\lambda \vec{r}}{4\pi r_{1/4}} \left[\frac{\Gamma(\theta) \cos(\Phi - \theta)\, d\theta}{\left[\lambda^2 + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)\right]^{3/2}} \right]$$

Integrating over the shroud:

A-10:

$$\vec{q}_{br} = \frac{\lambda \vec{r}}{4\pi r_{1/4}} \int_{-\Theta}^{\Theta} \frac{\Gamma(\theta) \cos(\Phi - \theta)\, d\theta}{\left[\lambda^2 + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)\right]^{3/2}}$$

Similarly:

A-11:

$$\vec{q}_{bx} = \frac{-\vec{i}}{4\pi r_{1/4}} \int_{-\Theta}^{\Theta} \frac{\Gamma(\theta) \left[\rho \cos(\Phi - \theta) - 1\right] d\theta}{\left[\lambda^2 + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)\right]^{3/2}}$$

$$dq_{b\Phi} = d\vec{q}_b \cdot \frac{d\vec{\Phi}}{|d\Phi|}$$

$$dq_{b\Phi}\vec{\varphi} = \left\{ \frac{-\Gamma(\theta)\, \lambda \cos\theta \sin\Phi \, d\theta}{4\pi r_{1/4} \left[\lambda^2 + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)\right]^{3/2}} \right.$$

$$\left. \frac{+\Gamma(\theta)\, \lambda \sin\theta \cos\varphi \, d\theta}{4\pi r_{1/4} \left[\lambda^2 + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)\right]^{3/2}} \right\} \vec{\varphi}$$

A-12:

$$\vec{q}_{b\Phi} = \frac{-\lambda\vec{\varphi}}{4\pi r_{1/4}} \int_{-\Theta}^{\Theta} \frac{\Gamma(\theta)\sin(\Phi-\theta)\,d\theta}{[\lambda^2+\rho^2+1-2\rho\cos(\Phi-\theta)]^{3/2}}$$

Note: $\vec{q}_{b\Phi}$ is of little interest in this problem and is included only for the sake of completeness.

IV. Integral Equations for the Velocity Distribution Induced by the Trailing Vortex Distribution:

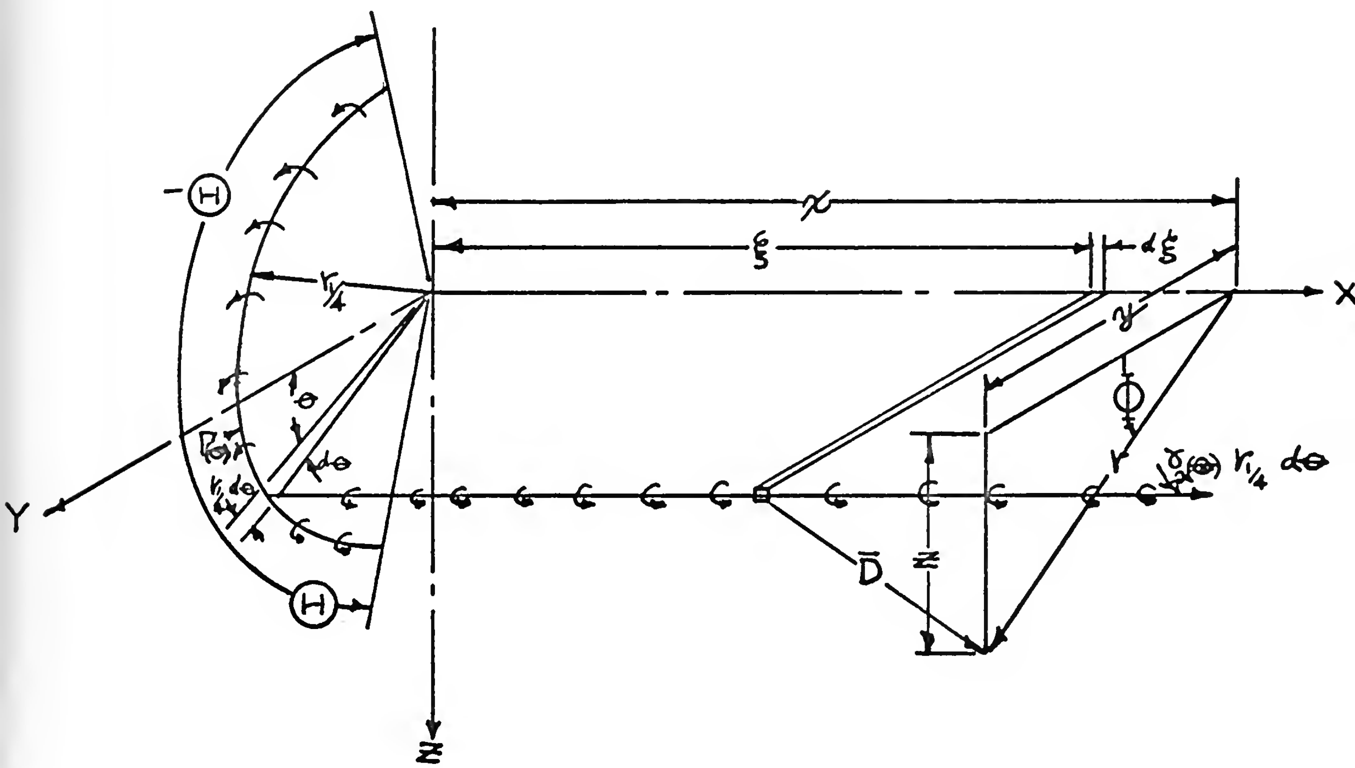


FIGURE A3

By A-1:

$$\gamma_x(\theta) = -\frac{1}{r_{1/4}} \frac{d\Gamma(\theta)}{d\theta}$$

By the law of Biot-Savart:

$$d\vec{q}_T = \frac{1}{4\pi}\left(-\frac{1}{r_{1/4}} \frac{d\Gamma(\theta)}{d\theta}\right) r_{1/4}\, d\theta\, \frac{d\vec{\xi} \times \vec{D}}{|D|^3}$$

A-13:

$$d\vec{q}_T = -\frac{1}{4\pi}\left(\frac{d\Gamma(\theta)}{d\theta}\right) d\theta\, \frac{d\vec{\xi} \times \vec{D}}{|D|^3}$$

$$D = (x-\xi)\vec{i} + (r\cos\Phi - r_{1/4}\cos\theta)\vec{j} + (r\sin\Phi - r_{1/4}\sin\theta)\vec{k}$$

$$d\vec{\xi} = d\xi\,\vec{i}$$

$$d\vec{\xi} \times \vec{D} = (0)\vec{i} - d\xi\,(r\sin\Phi - r_{1/4}\sin\theta)\vec{j} + d\xi\,(r\cos\Phi - r_{1/4}\cos\theta)\vec{k}$$

$$|D| = \left[(x-\xi)^2 + r^2 + r_{1/4}^2 - 2r\,r_{1/4}\cos(\Phi-\theta)\right]^{1/2}$$

Nondimensionalizing:

By A-7:

$$\frac{r}{r_{1/4}} = \rho\ ;\ \frac{x}{r_{1/4}} = \lambda$$

A-14:

Define $\frac{\xi}{r_{1/4}} = \epsilon$

A-15:

$$|D| = r_{1/4}\left[(\lambda - \epsilon)^2 + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)\right]^{1/2}.$$

A-16:

$$d\vec{\xi} \times \vec{D} = - r_{1/4}^2\left[d\epsilon\,(\rho \sin \Phi - \sin \theta)\vec{j} - d\epsilon\,(\rho \cos \Phi - \cos \theta)\,\vec{k}\right]$$

Substituting A-14, A-15, and A-16 into A-13:

$$d\vec{q}_T x = 0$$

$$\vec{d}q_T y = \frac{1}{4\pi r_{1/4}}\left(\frac{d\Gamma(\theta)}{d\theta}\right)\frac{(\rho \sin \Phi - \sin \theta)\, d\epsilon\, d\theta\, \vec{j}}{[(\lambda - \epsilon)^2 + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)]^{3/2}}$$

$$\vec{d}q_T z = - \frac{1}{4\pi r_{1/4}}\left(\frac{d\Gamma(\theta)}{d\theta}\right)\frac{(\rho \cos \Phi - \cos \theta)\, d\epsilon\, d\theta\, \vec{k}}{[(\lambda - \epsilon)^2 + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)]^{3/2}}$$

$$dq_{Tr} = d\vec{q}_T \bullet \frac{\vec{r}}{|r|} = d\vec{q}_T \cdot [\cos \Phi\, \vec{j} + \sin \Phi\, \vec{k}]$$

$$dq_{Tr}\vec{r} = \frac{1}{4\pi r_{1/4}}\ \frac{d\Gamma(\theta)}{d\theta}\ \frac{[+\sin(\Phi - \theta)]\, d\epsilon\, d\theta\, \vec{r}}{[(\lambda - \epsilon)^2 + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)]^{3/2}}$$

$$dq_{T\Phi} = d\vec{q}_T \bullet \frac{d\varphi}{|d\varphi|} = d\vec{q}_T \cdot [-\sin \Phi\, \vec{j} + \cos \Phi\, \vec{k}]$$

$$dq_{T\Phi}\vec{\varphi} = - \frac{1}{4\pi r_{1/4}}\ \frac{d\Gamma(\theta)}{d\theta}\ \frac{[P - \cos(\Phi - \theta)]\, d\epsilon\, d\theta\, \vec{\varphi}}{[(\lambda - \epsilon) + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)]^{3/2}}$$

integrating over the shroud:

A-17:

$$\vec{q}_{Tr} = \frac{\vec{r}}{4\pi r_{1/4}} \int_{-\Theta}^{\Theta}\int_{0}^{\infty} \frac{d\Gamma(\theta)}{d\theta}\ \frac{\sin(\Phi - \theta)\, d\epsilon\, d\theta}{[(\lambda - \epsilon)^2 + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)]^{3/2}}$$

A-18:

$$\vec{q}_{T\Phi} = -\frac{\vec{\varphi}}{4\pi r_{1/4}} \int_{-\Theta}^{\Theta}\int_{0}^{\infty} \frac{d\Gamma_{(\theta)}}{d\theta} \frac{[\rho - \cos(\Phi-\theta)]\, d\epsilon\, d\theta}{[(\lambda-\epsilon)^2 + \rho^2 + 1 - 2\rho\cos(\Phi-\theta)]^{3/2}}$$

A-19:

$$\vec{q}_{Tx} = 0$$

Considering A-17:

$$\vec{q}_{Tr} = \frac{\vec{r}}{4\pi r_{1/4}} \int_{-\Theta}^{\Theta} \frac{d\Gamma_{(\theta)}}{d\theta} \sin(\Phi-\theta) \left\{ \frac{d\epsilon}{[(\lambda-\epsilon)^2 + k^2]^{3/2}} \right\} d\theta$$

where $k^2 \equiv \rho^2 + 1 - 2\rho\cos(\Phi-\theta)$ can be treated as constant when integrating the integral in brackets ($k^2 \geq 0$). Integrating the bracketed integral by parts:

Let $u = \lambda - \epsilon; du = -d\epsilon$

LIMITS: $\epsilon = 0 \Rightarrow u = \lambda$

$\epsilon = \infty \Rightarrow u = -\infty$

$$\int_0^{\infty} \frac{d\epsilon}{[(\lambda-\epsilon)^2 + k^2)^{3/2}} = \int_{\lambda}^{-\infty} -\frac{du}{[u^2 + k^2]^{3/2}} = \frac{u}{k^2(u^2+k^2)^{1/2}}\Bigg|_{-\infty}^{\lambda}$$

$$= \frac{\lambda}{k^2(\lambda^2+k^2)^{1/2}} + \frac{1}{k^2}$$

A-20:

$$\vec{q}_{Tr} = \frac{\vec{r}\lambda}{4\pi r_{1/4}} \int_{-\Theta}^{\Theta} \frac{d\Gamma_{(\theta)}}{d\theta} \frac{\sin(\Phi-\theta)\, d\theta}{[\rho^2 + 1 - 2\rho\cos(\Phi-\theta)]\, [\lambda^2 + \rho^2 + 1 - 2\rho\cos(\Phi-\theta)]^{1/2}}$$

$$+ \frac{\vec{r}}{4\pi r_{1/4}} \int_{-\Theta}^{\Theta} \frac{d\Gamma_{(\theta)}}{d\theta} \frac{\sin(\Phi-\theta)\, d\theta}{[\rho^2 + 1 - 2\rho\cos(\Phi-\theta)]}$$

V. Complete Integral Equations for the Velocity Induced by the Shroud:

Combining A-10 and A-20:

$$\vec{q}_{r_s} = \vec{q}_{br} + \vec{q}_{Tr}$$

A-21:

$$\vec{q}_{r_s} = \frac{\lambda \vec{r}}{4\pi r_{1/4}} \int_{-\Theta}^{\Theta} \frac{\Gamma(\theta) \cos(\Phi - \theta)\, d\theta}{[\lambda^2 + \rho^2 + 1 - 2\rho \cos(\Phi - \theta)]^{3/2}}$$

$$+ \frac{\lambda \vec{r}}{4\pi r_{1/4}} \oint_{-\Theta}^{\Theta} \frac{d\Gamma_{(\theta)}}{d\theta} \frac{\sin(\varphi - \theta)\, d\theta}{[\rho^2 + 1 - 2\rho\cos(\Phi - \theta)]\,[\lambda^2 + \rho^2 + 1 - 2\rho\cos(\Phi - \theta)]^{1/2}}$$

$$+ \frac{\vec{r}}{4\pi r_{1/4}} \oint_{-\Theta}^{\Theta} \frac{d\Gamma_{(\theta)}}{d\theta} \frac{\sin(\Phi - \theta)\, d\theta}{[\rho^2 + 1 - 2\rho\cos(\Phi - \theta)]}$$

By A-11:

$$\vec{q}_{xs} = \vec{q}_{bx} = -\frac{i}{4\pi r_{1/4}} \int_{-\Theta}^{\Theta} \frac{\Gamma(\theta)[\rho\cos(\Phi - \theta) - 1]\, d\theta}{[\lambda^2 + \rho^2 + 1 - 2\rho\cos(\Phi - \theta)]^{3/2}}$$

APPENDIX B

SOLUTION OF THE INTEGRAL EQUATIONS FOR A PIVOTAL POINT COINCIDENT WITH THE TRAILING VORTEX SHEET

I. Reduction of Eq. (A-21) to Inegrable Form

For a pivotal point on the trailing vortex sheet, $r = r_{1/4}$, $\rho = 1$, and the integrals involving $d\Gamma_{(\theta)}/d\theta$ which arise from the trailing vortices must be evaluated as Cauchy Principle Values.

A-22:

$$\vec{q}_{r_s} = \frac{\lambda \vec{r}}{4\pi r_{1/4}} \int_{-\Theta}^{\Theta} \frac{\Gamma_{(\theta)} \cos(\Phi - \theta)\, d\theta}{[\lambda^2 + 2 - 2\cos(\Phi - \theta)]^{3/2}}$$

$$+ \frac{\lambda \vec{r}}{8\pi r_{1/4}} \oint_{-\Theta}^{\Theta} \frac{d\Gamma_{(\theta)}}{d\theta} \frac{\sin(\Phi - \theta)\, d\theta}{[1 - \cos(\Phi - \theta)]\,[\lambda^2 + 2 - 2\cos(\Phi - \theta)]^{1/2}}$$

$$+ \frac{\vec{r}}{8\pi r_{1/4}} \oint_{-\Theta}^{\Theta} \frac{d\Gamma_{(\theta)}}{d\theta} \frac{\sin(\Phi - \theta)\, d\theta}{[1 - \cos(\Phi - \theta)]}$$

Assuming that the circulation distribution can be represented as a complete Fourier series:

$$\Gamma(\theta) = \sum_{n=o}^{\infty} A_n \cos(n\theta) + B_n \sin(n\theta)$$

The boundary conditions on this representation are:

a. $\Gamma(\theta)$ is an even function of θ for a shroud symmetrical about the $\theta = 0$ plane.

b. $\Gamma(\Theta) = \Gamma(-\Theta) = 0$ $(\Theta \neq \pi)$; that is the circulation goes to zero at the tips of the shroud.

By (a) $B_n = 0$ for all n.

A-23:

$$\Gamma(\theta) = \sum_{n=o}^{\infty} A_n \cos(n\theta)$$

$$\frac{d\Gamma(\theta)}{d\theta} = - \sum_{n=o}^{\infty} n\, A_n \sin(n\theta)$$

A-24:

Define $\frac{\lambda^2 + 2}{2} = \ell$

Substituting A-23 and A-24 into A-22 and summing over the integrals:

A-25:

$$\vec{q}_{r_s} = \frac{\vec{r}\lambda \sum_{n=o}^{\infty} A_n}{8\sqrt{2}\pi r_{1/4}} \int_{-\Theta}^{\Theta} \frac{\cos(n\theta)\cos(\Phi - \theta)\, d\theta}{[\ell - \cos(\Phi - \theta)]^{3/2}}$$

$$- \frac{\vec{r}\lambda \sum_{n=o}^{\infty} A_n}{8\sqrt{2}\pi r_{1/4}} \oint_{-\Theta}^{\Theta} \frac{\sin(n\theta)\sin(\Phi - \theta)\, d\theta}{[1 - \cos(\Phi - \theta)]\,[\ell - \cos(\Phi - \theta)]^{1/2}}$$

$$- \frac{\vec{r} \sum_{n=o}^{\infty} n A_n}{8\pi r_{1/4}} \oint_{-\Theta}^{\Theta} \sin(n\theta) \frac{\sin(\Phi - \theta)\, d\theta}{[1 - \cos(\Phi - \theta)]}$$

Transformation of variables:

$$n\theta = n\Phi - n(\Phi - \theta)$$

$$\cos n\theta = \cos n(\Phi - \theta)\cos n\varphi + \sin n(\Phi - \theta)\sin n\varphi$$

$$\sin n\theta = \cos n(\Phi - \theta)\sin n\varphi - \sin n(\Phi - \theta)\cos n\varphi$$

Define:

$$\nu \equiv \pi - (\Phi - \theta) \; ; \; d\nu = d\theta$$

$$\sin(\Phi - \theta) = \sin \nu$$

$$\cos(\Phi - \theta) = -\cos \nu$$

$$\cos n\theta = \cos n(\pi - \nu) \cos n\Phi + \sin n(\pi - \nu) \sin n\Phi$$

$$\sin n\theta = \cos n(\pi - \nu) \sin n\Phi - \sin n(\pi - \nu) \cos n\Phi$$

since

$$n(\pi - \nu) = n\pi - n\nu \; ;$$

$$\cos(n\pi) = (-1)^n$$

$$\sin(n\pi) = (0)^m,$$

$$\cos n(\pi - \nu) = (-1)^n \cos n\nu + (0)^n \sin n\nu = (-1)^n \cos n\nu$$

$$\sin n(\pi - \nu) = (0)^n \cos n\nu - (-1)^n \sin n\nu = -(-1)^n \sin n\nu$$

$$\therefore \cos n\theta = (-1)^n [\cos n\nu \cos n\Phi - \sin n\nu \sin n\Phi]$$

$$\sin n\theta = (-1)^n [\cos n\nu \sin n\Phi + \sin n\nu \cos n\Phi]$$

Revised limits of integration:

when

$$\theta = +\Theta \; ; \; \nu = \pi - (\Phi - \Theta) = \pi + \Theta - \Phi \equiv \nu_2$$

when

$$\theta = -\Theta \; ; \; \nu = \pi - (\Phi + \Theta) = \pi - \Theta - \Phi \equiv \nu_1$$

Note: Restriction: $\nu_1, \nu_2 \neq \pi \quad \Phi \neq \pm\Theta$

Further defining:

$$C_{n1} \equiv (-1)^{n+1} \frac{\lambda \cos(n\Phi) A_n}{8\sqrt{2}\, \pi\, r_{1/4}} \qquad I_{n1} \equiv \int_{\nu_1}^{\nu_2} \frac{\cos(n\nu) \cos \nu \, d\nu}{[\ell + \cos \nu]^{3/2}}$$

$$C_{n2} \equiv (-1)^{n} \frac{\lambda \sin (n\Phi) A_n}{8\sqrt{2}\pi r_{1/4}} \qquad I_{n2} \equiv \int_{\nu_1}^{\nu_2} \frac{\sin (n\nu) \cos \nu \, d\nu}{[\ell + \cos \nu]^{3/2}}$$

$$C_{n3} \equiv (-1)^{n+1} \frac{\lambda n \sin (n\Phi) A_n}{8\sqrt{2}\pi r_{1/4}} \qquad I_{n3} \equiv \int_{\nu_1}^{\nu_2} \frac{\cos (n\nu) \sin \nu \, d\nu}{[1+\cos \nu][\ell+\cos]^{1/2}}$$

$$C_{n4} \equiv (-1)^{n+1} \frac{\lambda n \cos (n\Phi) A_n}{8\sqrt{2}\pi r_{1/4}} \qquad I_{n4} \equiv \int_{\nu_1}^{\nu_2} \frac{\sin (n\nu) \sin \nu \, d\nu}{[1+\cos \nu][\ell + \cos \nu]^{1/2}}$$

$$C_{n5} \equiv (-1)^{n+1} \frac{n \sin (n\Phi) A_n}{8\pi r_{1/4}} \qquad I_{n5} \equiv \int_{\nu_1}^{\nu_2} \frac{\cos (n\nu) \sin \nu \, d\nu}{[1+\cos \nu]}$$

$$C_{n6} \equiv (-1)^{n+1} \frac{n \cos (n\Phi) A_n}{8\pi r_{1/4}} \qquad I_{n6} \equiv \int_{\nu_1}^{\nu_2} \frac{\sin (n\nu) \sin \nu \, d\nu}{[1+\cos \nu]}$$

Based on the above definitions A-25 reduces to:

A-26:

$$\vec{q}_{r_s} = \vec{r} \sum_{n=o}^{\infty} \sum_{m=1}^{6} C_{nm} I_{nm}$$

Solutions to the above integrals are given in the following equations: $(n \leq 3)$

	Integral	Eq.		Integral	Eq.
m = 1:	I_{01}	A-43	m = 4:	I_{04}	A-47
	I_{11}	A-44		I_{14}	A-48
	I_{21}	A-45		I_{24}	A-49
	I_{31}	A-46		I_{34}	A-50
m = 2:	I_{02}	A-27	m = 5:	I_{05}	A-35
	I_{12}	A-28		I_{15}	A-36
	I_{22}	A-29		I_{25}	A-37
	I_{32}	A-30		I_{35}	A-38
m = 3:	I_{03}	A-31	m = 6:	I_{06}	A-39
	I_{13}	A-32		I_{16}	A-40
	I_{23}	A-33		I_{26}	A-41
	I_{33}	A-34		I_{36}	A-42

Note: In evaluating the integrals I_{n1} and I_{n4} which involve elliptic functions the following relations are used:

$$k^2 = \frac{2}{\ell+1} \; ; \; k'^2 = 1 - k^2 = \frac{\ell - 1}{\ell+1}$$

$$g = \frac{2}{\sqrt{\ell+1}} \; ; \; \frac{g}{k^2} = \sqrt{\ell+1} \; ; \; 1 + k'^2 = \frac{2\ell}{\ell+1}$$

$$\varphi_p = \mathrm{am}\, u_p = \sin^{-1} \sqrt{\frac{(\ell+1)(1+\cos \nu_p)}{2(\ell + \cos \nu_p)}}$$

Note: See Figure (A-5) for quadrant selection.

$$\mathrm{sn}(u_p, k) = \sin \varphi_p = \mathrm{sn}\, u_p \; ; \; \mathrm{cn}\, u_p = \cos \varphi_p$$

$$\mathrm{dn}\, u = \sqrt{1 - k^2 \sin^2 \varphi}$$

$$\mathrm{ns}\, u = \frac{1}{\mathrm{sn}\, u} \qquad \mathrm{tn}\, u = \mathrm{sc}\, u = \frac{\mathrm{sn}\, u}{\mathrm{cn}\, u} \qquad \mathrm{sd}\, u = \frac{\mathrm{sn}\, u}{\mathrm{dn}\, u}$$

$$\mathrm{nc}\, u = \frac{1}{\mathrm{cn}\, u} \qquad \mathrm{cs}\, u = \frac{\mathrm{cn}\, u}{\mathrm{sn}\, u} \qquad \mathrm{cd}\, u = \frac{\mathrm{cn}\, u}{\mathrm{dn}\, u}$$

$$\mathrm{nd}\, u = \frac{1}{\mathrm{dn}\, u} \qquad \mathrm{ds}\, u = \frac{\mathrm{dn}\, u}{\mathrm{sn}\, u} \qquad \mathrm{dc}\, u = \frac{\mathrm{dn}\, u}{\mathrm{cn}\, u}$$

$$V_1(u) = \frac{1}{k'^2} \left[E(u) - k^2 \,\mathrm{sn}\, u \,\mathrm{cd}\, u \right]$$

$$V_2(u) = \frac{1}{3k'^4} \left[2(2 - k^2)\, E(u) - k'^2 u - k^2 \,\mathrm{sn}\, u \,\mathrm{cd}\, u \,(k'^2 \,\mathrm{nd}^2 u + 4 - 2k^2) \right]$$

$$V_3(u) = \frac{4(2 - k^2)\, V_2 - 3V_1 - k^2 \,\mathrm{sn}\, u \,\mathrm{cd}\, u \,\mathrm{nd}^4 u}{5k'^2}$$

<u>Summary of Integral Values:</u>

$$I_{01} = \frac{2}{\sqrt{\ell+1}} \left\{ u_1 - u_2 - \frac{\ell}{\ell-1} [E(u_1) - E(u_2)] \right\} \tag{A-43}$$

$$I_{11} = \frac{2(\ell-1)}{\sqrt{\ell+1}} \left[\pi(\varphi_1, k^2, k) - \pi(\varphi_2, k^2, k) \right] - \frac{2\ell}{\sqrt{\ell+1}} (u_1 - u_2) - \ell[I_{01}] \tag{A-44}$$

$$I_{21} = 2\ell^2(u_1 - u_2) + 4\ell(1-\ell)[V_1(u_1) - V_1(u_2)] + 2(1-\ell)^2 [V_2(u_1) - v_2(u_2)]$$
$$- 2\ell[I_{11}] - [I_{01}] \tag{A-45}$$

$$I_{31} = 4\ell^3(u_1 - u_2) + 12\ell^2(1-\ell)[V_1(u_1) - V_1(u_2)] + 12\ell(1-\ell)^2 [V_2(u_1) - V_2(u_2)]$$
$$+4(1-\ell)^3 [V_3(u_1) - V_3(u_2)]$$
$$-2\ell[I_{21}] - 3[I_{11}] - 2\ell[I_{01}] \tag{A-46}$$

$$I_{02} = 0 \tag{A-27}$$

$$I_{12} = 2\left[(\ell+t)^{\frac{1}{2}} + \ell(\ell+t)^{-\frac{1}{2}} \right]_{\cos \nu_2}^{\cos \nu_1} \tag{A-28}$$

$$I_{22} = 4\left\{ \frac{-t^2}{(\ell+t)^{\frac{1}{2}}} + 4\left[\frac{(\ell+t)^{3/2}}{3} - \ell(\ell+t)^{\frac{1}{2}} \right] \right\} \Bigg|_{\cos \nu_2}^{\cos \nu_1} \tag{A-29}$$

$$I_{32} = \left\{ \frac{16}{5}\left(3t^2 - 4\ell t + 8\ell^2 - \frac{5}{8} \right) (\ell+t)^{\frac{1}{2}} - 8(t^3 + 2\ell)(\ell+t)^{-\frac{1}{2}} \right\}_{\cos \nu_2}^{\cos \nu_1} \tag{A-30}$$

$$I_{03} = \frac{1}{\sqrt{\ell-1}} \left\{ \ln \left[\frac{\sqrt{\ell+t} - \sqrt{\ell-1}}{\sqrt{\ell+t} + \sqrt{\ell-1}} \right] \right\} \Bigg|_{\cos \nu_2}^{\cos \nu_1} \tag{A-31}$$

$$I_{13} = 2\sqrt{\ell+t}\,\Big|_{\cos\nu_2}^{\cos\nu_1} - [I_{03}] \tag{A-32}$$

$$I_{23} = \left(\frac{4t}{3} - \frac{8\ell}{3} - 4\right)\sqrt{\ell+t}\,\Big|_{\cos\nu_2}^{\cos\nu_1} + [I_{03}] \tag{A-33}$$

$$I_{33} = \frac{8}{15}\left[3t^2 - 5t - 4\ell t + 8\ell^2 + 10\ell + \frac{15}{4}\right]\sqrt{\ell+t}\,\Big|_{\cos\nu_2}^{\cos\nu_1} - [I_{03}] \tag{A-34}$$

$$I_{04} = 0 \tag{A-47}$$

$$I_{14} = 2\sqrt{\ell+1}\left[u_1 - u_2 + E(u_2) - E(u_1) + \frac{2}{\ell+1}\,(\mathrm{sn}\,u_1\,\mathrm{cd}\,u_1 - \mathrm{sn}\,u_2\,\mathrm{cd}\,u_2)\right] \tag{A-48}$$

$$\begin{aligned} I_{24} = {} & \frac{4}{3}(\ell-1)\sqrt{\ell+1}\,\Big\{2u_2 - 2u_1 + \frac{2\ell}{\ell-1}[E(u_1) - E(u_2)] \\ & + \frac{2}{\ell+1}\left[\left(\mathrm{nd}^2 u_1 - \frac{2\ell}{\ell-1}\right)\mathrm{sn}\,u_1\,\mathrm{cd}\,u_1 - \left(\mathrm{nd}^2 u_2 - \frac{2\ell}{\ell-1}\right)\times \right. \\ & \left. \mathrm{sn}\,u_2\,\mathrm{cd}\,u_2\right]\Big\} - 2[I_{14}] \end{aligned} \tag{A-49}$$

$$\begin{aligned} I_{34} = {} & 8(\ell-1)^2\sqrt{\ell+1}\,\Big\{V_1(u_2) - V_1(u_1) + \frac{2\ell}{\ell+1}\left[V_2(u_1) - V_2(u_2)\right] \\ & + \frac{\ell-1}{\ell+1}\left[V_3(u_2) - V_3(u_1)\right]\Big\} - 2(\ell+1)[I_{24}] - (4\ell+1)[I_{14}] \end{aligned} \tag{A-50}$$

$$I_{05} = \ell n\left[\frac{1+\cos\nu_1}{1+\cos\nu_2}\right] \tag{A-35}$$

$$I_{15} = \cos\nu_1 - \cos\nu_2 - [I_{05}] \tag{A-36}$$

$$I_{25} = \cos^2\nu_1 - \cos^2\nu_2 - 2[I_{15}] - [I_{05}] \tag{A-37}$$

$$I_{35} = \frac{4}{3}\left(\cos^3\nu_1 - \cos^3\nu_2\right) - 2[I_{25}] - 3[I_{15}] - 4[I_{05}] \tag{A-38}$$

$$I_{06} = 0 \tag{A-39}$$

$$I_{16} = \sin \nu_1 - \sin \nu_2 - \nu_1 + \nu_2 \tag{A-40}$$

$$I_{26} = 2 \sin \nu_2 - 2 \sin \nu_1 + \tfrac{1}{2} (\sin 2\nu_1 - \sin 2\nu_2) - (\pm \nu_1) + (\pm \nu_2) \tag{A-41}$$

Note: Sign is negative if ν_p lies in the first or second quadrant and positive for third and fourth quadrants.

$$I_{36} = \frac{4}{3}\left(\sin^3 \nu_2 - \sin^3 \nu_1\right) - 2[I_{26}] - [I_{16}] \tag{A-42}$$

II. SOLUTION OF THE GEOMETRIC INTEGRALS, I_{nm} IN EQUATION A-26

Integrals I_{n1} and I_{n4} are elliptic integrals and are most readily solved in terms of Jacobian Elliptic Functions. We will defer this step until last and first consider integrals I_{n2}, I_{n3}, I_{n5}, and I_{n6} which can be reduced to elementary forms.

Using the algebraic transformation:

$$\cos \nu = t \;;\; \sin \nu = \sqrt{1-t^2}$$

$$-\sin \nu \, d\nu = dt$$

$$d\nu = -\frac{dt}{\sqrt{1-t^2}}$$

n	$\cos n\nu \equiv g_n(t)$	$\sin n\nu = f_n(t)$
0	1	0
1	t	$\sqrt{(1-t^2)}$
2	$2t^2 - 1$	$2t\sqrt{(1-t^2)}$
3	$4t^3 - 3t$	$3\sqrt{(1-t^2)}$ $-4(1-t^2)\sqrt{(1-t^2)}$ $=(4t^2-1)\sqrt{(1-t^2)}$

Considering I_{n2}:

$$I_{n2} \equiv \int_{\nu_1}^{\nu_2} \frac{\sin n\nu \cos \nu \, d\nu}{[\ell + \cos \nu]^{3/2}}$$

$$= \int_{\cos \nu_2}^{\cos \nu_1} \frac{\nu_n(t)\; t\, dt}{(1-t^2)^{1/2}\,[\ell+t]^{3/2}}$$

(Note reversal of limit with sign change)

<u>For n = 0:</u> $f_n(t) = 0$

<u>A-27:</u> $I_{02} = 0$

<u>For n = 1:</u> $f_n(t) = \sqrt{1 - t^2}$

$$I_{12} = + \int_{\cos \nu_2}^{\cos \nu_1} \frac{t \, dt}{[\ell + t]^{3/2}}$$

By 119, Ref. (3):

$$\int x \, (a+bx)^{\pm n/2} \, dx = \frac{2}{b^2} \left[\frac{(a+bx)^{(4 \pm n)/2}}{4 \pm n} - \frac{a(a+bx)^{(2 \pm n)/2}}{2 \pm n} \right]$$

with $a = \ell$, $b = 1$, $\pm n = -3$.

<u>A-28:</u> $I_{12} = 2 \left[(\ell + t)^{1/2} + \ell(\ell + t)^{-1/2} \right] \Big|_{\cos \nu_2}^{\cos \nu_1}$

<u>For N = 2:</u> $f_n(t) = 2t \sqrt{(1 - t^2)}$

$$I_{22} = 2 \int_{\cos \nu_2}^{\cos \nu_1} \frac{t^2 \, dt}{[\ell + t]^{3/2}}$$

Integrating by parts with $u = t^2$, $du = 2t \, dt$

$$dv = \frac{dt}{(\ell + t)^{3/2}} \qquad v = \frac{-2}{(\ell + t)^{1/2}}$$

$$I_{22} = 2 \left[- \frac{2t^2}{(\ell + t)^{1/2}} \Bigg|_{\cos \nu_2}^{\cos \nu_1} + 4 \int_{\cos \nu_2}^{\cos \nu_1} \frac{t \, dt}{(\ell + t)^{1/2}} \right]$$

By 119, Ref. (3): with $a = \ell$, $b = 1$, $\pm n = -1$

A-29: $$I_{22} = 4\left\{\frac{-t^2}{(\ell+t)^{1/2}} + 4\left[\frac{(\ell+t)^{3/2}}{3} - \ell(\ell+t)^{1/2}\right]\right\}\Bigg|_{\cos\nu_2}^{\cos\nu_1}$$

For n = 3: $f_n(t) = (4t^2 - 1)\sqrt{1 - t^2}$

$$I_{32} = 4\int_{\cos\nu_2}^{\cos\nu_1} \frac{t^3\,dt}{[\ell+t]^{3/2}} - \int_{\cos\nu_2}^{\cos\nu_1} \frac{t\,dt}{[\ell+t]^{3/2}}$$

Integrating the first by parts with: $u = t^3$, $du = 3t^2\,dt$

$$dv = \frac{dt}{(\ell+t)^{3/2}} \qquad v = \frac{-2}{(\ell+t)^{1/2}}$$

The first integral equals

$$4\left\{-\frac{2t^3}{(\ell+t)^{1/2}}\Bigg|_{\cos\nu_2}^{\cos\nu_1} + 6\int_{\cos\nu_2}^{\cos\nu_1} \frac{t^2\,dt}{(\ell+t)^{1/2}}\right\}$$

By 112, Ref. (3):

$$\int \frac{x^2\,dx}{\sqrt{a+bx}} = \frac{2(8a^2 - 4abx + 3b^2x^2)}{15b^3}\sqrt{a+bx}$$

with $a = \ell$, $b = 1$:

$$\left\{-\frac{8t^3}{(\ell+t)^{1/2}} + \frac{48}{15}(8\ell^2 - 4\ell t + 3t^2)(\ell+t)^{1/2}\right\}\Bigg|_{\cos\nu_2}^{\cos\nu_1}$$

The second integral is just A-28 ∴ combining:

A-30:

$$I_{32} = \left\{\frac{16}{5}(3t^2 - 4\ell t + 8\ell^2 - \frac{5}{8})(\ell+t)^{1/2} - (8t^3 + 2\ell)(\ell+t)^{-1/2}\right\}\Bigg|_{\cos\nu_2}^{\cos\nu_1}$$

Considering I_{n3}:

$$I_{n3} = \int_{\nu_1}^{\nu_2} \frac{\cos n\nu \sin\nu \, d\nu}{[1+\cos \nu]\,[\ell + \cos \nu]^{1/2}}$$

$$= \int_{\cos \nu_2}^{\cos \nu_1} \frac{g_n(t)\, dt}{[1+t]\,[\ell + \;]^{1/2}}$$

For n = 0: $g_n(t) = 1$

$$I_{03} = \int_{\cos \nu_2}^{\cos \nu_1} \frac{dt}{[1+t]\,[\ell + t]^{1/2}}$$

By 56, Ref. (3):

$$\int \frac{dx}{v\sqrt{u}} = \frac{1}{\sqrt{kb'}}\, \ell n\, \frac{b'\sqrt{u} - \sqrt{kb'}}{b'\sqrt{u} + \sqrt{kb'}}$$

with

$$v = 1 + t = a' + b't \;; \quad a' = 1b' = 1$$

$$u = \ell + t = a + bt; \quad a = \ell, b = 1$$

$$k = ab' - a'b = (\ell - 1)$$

A-31:

$$I_{03} = \frac{1}{\sqrt{\ell - 1}} \left\{ \ell n \left[\frac{\sqrt{\ell + t} - \sqrt{\ell - 1}}{\sqrt{\ell + t} + \sqrt{\ell - 1}} \right] \right\}_{\cos\ \ 2}^{\cos \nu_1}$$

For n = 1: $g_n(t) = t$

$$I_{13} = \int_{\cos \nu_2}^{\cos \nu_1} \frac{t\, dt}{[1+t]\,[\ell + t]^{1/2}}$$

Using v and u as defined in I_{03}, $t = v - 1$

$$I_{13} = \int_{\cos \nu_2}^{\cos \nu_1} \frac{(v-1)dt}{v\sqrt{u}} = \int_{\cos \nu_2}^{\cos \nu_1} \frac{dt}{\sqrt{u}} - \int_{\cos \nu_2}^{\cos \nu_1} \frac{dt}{v\sqrt{u}}$$

$$= \left\{ 2\sqrt{\ell + t} - \frac{-1}{\sqrt{\ell - 1}} \ell n \left[\frac{\sqrt{\ell+t} - \sqrt{\ell - 1}}{\sqrt{\ell+t} + \sqrt{\ell - 1}} \right] \right\} \Bigg|_{\cos \nu_2}^{\cos \nu_1}$$

<u>A-32:</u>

$$I_{13} = 2\sqrt{\ell+t} \Bigg|_{\cos \nu_2}^{\cos \nu_1} - \left[I_{03} \right]$$

<u>For n = 2:</u> $g_n(t) = 2t^2 - 1 = 2(v-1)^2$

$g_n(t) = 2v^2 - 4v + 1$

$$I_{23} = 2 \int_{\cos \nu_2}^{\cos \nu_1} \frac{v\, dt}{\sqrt{u}} - 4 \int_{\cos \nu_2}^{\cos \nu_1} \frac{dt}{\sqrt{u}} + \int_{\cos \nu_2}^{\cos \nu_1} \frac{dt}{v\sqrt{u}}$$

since $u = \ell + t = 1 + t - (1 - \ell) = v - (1 - \ell)$

$dt = dv$

the first integral $= 2 \int_{1+\cos \nu_2}^{1+\cos \nu_1} \frac{v\, dv}{\sqrt{(\ell - 1) + v}}$

By 111, Ref. (3):

$$2 \int \frac{x\, dx}{\sqrt{a+bx}} = - \frac{2(2a - bx)}{3b^2} \sqrt{a + bx}$$

with: $x = v$, $a = (\ell - 1)$, $b = 1$

the first integral $= - \frac{4}{3} (2\ell - 2 - v) \sqrt{(\ell - 1) + v} \Bigg|_{1+\cos \nu_2}^{1+\cos \nu_1}$

reverting back to t:

$$\Rightarrow -\frac{4}{3}(2\ell - 3 - t)\sqrt{\ell+t}\,\Big|_{\cos\nu_2}^{\cos\nu_1}$$

the second integral $= -8\sqrt{\ell+t}\,\Big|_{\cos\nu_2}^{\cos\nu_1}$

the third integral, from $A_{30} = \dfrac{1}{\sqrt{\ell-1}}\left\{\ell n\left[\dfrac{\sqrt{\ell+t}-\sqrt{\ell-1}}{\sqrt{\ell+t}+\sqrt{\ell-1}}\right]\right\}\Big|_{\cos\nu_2}^{\cos\nu_1}$

Combining:

$$I_{23} = \left(\frac{4t}{3} - \frac{8\ell}{3} - 4\right)\sqrt{\ell+t} + \frac{1}{\sqrt{\ell-1}}\left\{\ell n\left[\frac{\sqrt{\ell+t}-\sqrt{\ell-1}}{\sqrt{\ell+t}+\sqrt{\ell-1}}\right]\right\}\Big|_{\cos\nu_2}^{\cos\nu_1}$$

<u>A-33:</u>

$$I_{23} = \left(\frac{4t}{3} - \frac{8\ell}{3} - 4\right)\sqrt{\ell+t}\,\Big|_{\cos\nu_2}^{\cos\nu_1} + \left[I_{03}\right]$$

<u>For n = 3:</u> $g_n(t) = 4t^3 - 3t$

$$= 4(v-1)^3 - 3(v-1)$$

$$g_n(t) = 4[v^3 - 3v^2 + 3v - 1] - 3v + 3$$

$$= 4v^3 - 12v^2 + 9v - 1$$

By the procedure followed in solving I_{23}:

$$I_{33} = 4\int_{1+\cos\nu_2}^{1+\cos\nu_1}\frac{v^2 dv}{\sqrt{(\ell-1)+v}} - 12\int_{1+\cos\nu_2}^{1+\cos\nu_1}\frac{v\,dv}{\sqrt{(\ell-1)+v}}$$

$$+ 9\int_{\cos\nu_2}^{\cos\nu_1}\frac{dt}{\sqrt{u}} - \int_{\cos\nu_2}^{\cos\nu_1}\frac{dt}{v\sqrt{u}}$$

By 112, Ref. (3):

$$\int \frac{x^2\,dx}{\sqrt{a+bx}} = \frac{2(8a^2 - 4\,abx + 3b^2x^2)}{15b^3}\sqrt{a+bx}$$

with x = v, a = ℓ - 1, b = 1, the first integral yields

$$8\left[\frac{8(\ell-1)^2 - 4(\ell-1)v + 3\,v^2}{15}\right]\sqrt{(\ell-1)+v}\,\Bigg|_{1+\cos\nu_2}^{1+\cos\nu_1}$$

reverting back to t:

$$\Rightarrow \frac{8}{15}\left[8\ell^2 - 16\ell + 8 - 4\ell + 4 - 4\ell t + 4t + 3 + 3t^2 + 6t\right]\sqrt{\ell+t}\,\Bigg|_{\cos\nu_2}^{\cos\nu_1}$$

$$= \frac{8}{15}\left[3t^2 + 10t - 4\,\ell t + 8\ell^2 - 20\ell + 15\right]\sqrt{\ell+t}\,\Bigg|_{\cos\nu_2}^{\cos\nu_1}$$

The second integral yields, from the development of A-33:

$$8(2\ell - 3 - t)\,\sqrt{\ell+t}\,\Bigg|_{\cos\nu_2}^{\cos\nu_1}$$

The third integral yields:

$$18\sqrt{\ell+t}\,\Bigg|_{\cos\nu_2}^{\cos\nu_1}$$

The fourth integral, from A-30 yields:

$$-\frac{1}{\sqrt{\ell-1}}\;\ell n\;\frac{\sqrt{\ell+t} - \sqrt{\ell-1}}{\sqrt{\ell+t} + \sqrt{\ell-1}}\,\Bigg|_{\cos\nu_2}^{\cos\nu_1}$$

combining:

$$I_{33} = \left\{ \frac{8}{15} \left[3t^2 + 10t - 15t - 4\,\ell t + 8\ell^2 - 20\ell + 30\ell + 15 - 45 + \frac{135}{4}\right]\right.$$

$$\left.\sqrt{\ell+t} - \frac{1}{\sqrt{\ell-1}}\ \ell n \left[\frac{\sqrt{\ell+t} - \sqrt{\ell-1}}{\sqrt{\ell+t} + \sqrt{\ell-1}}\right]\right\} \Bigg|_{\cos\nu_2}^{\cos\nu_1}$$

$$= \left\{ \frac{8}{15} \left[3t^2 - 5t - 4\,\ell t + 8\ell^2 + 10\ell + \frac{15}{4}\right] \sqrt{\ell+t} - \frac{1}{\sqrt{\ell-1}}\right.$$

$$\left.\ell n \left[\frac{\sqrt{\ell+t} - \sqrt{\ell-1}}{\sqrt{\ell+t} + \sqrt{\ell-1}}\right]\right\} \Bigg|_{\cos\nu_2}^{\cos\nu_1}$$

A-34:

$$I_{33} = \frac{8}{15}\left(3t^2 - 5t - 4\,\ell t + 8\,\ell^2 + 10\,\ell + \frac{15}{4}\right)\sqrt{\ell+t}\,\Bigg|_{\cos\nu_2}^{\cos\nu_1} - [I_{03}]$$

Considering I_{n5}:

$$I_{n5} \equiv \int_{\nu_1}^{\nu_2} \frac{\cos n\nu \sin\nu\, d\nu}{[1+\cos\nu]} = \int_{\cos\nu_2}^{\cos\nu_1} \frac{g_n(t)\, dt}{[1+t]}$$

For n = 0: $g_n(t) = 1$

$$I_{05} = \int_{\cos\nu_2}^{\cos\nu_1} \frac{dt}{[1+t]} = \ell n(1+t)\Bigg|_{\cos\nu_2}^{\cos\nu_1}$$

A-35:

$$I_{05} = \ell n\left[\frac{1+\cos\nu_1}{1+\cos\nu_2}\right]$$

For n = 1: $g_n(t) = t$

$$I_{15} = \int_{\cos \nu_2}^{\cos \nu_1} \frac{t\, dt}{[1+t]}$$

By 32 Ref. (3)

$$\int \frac{x\, dx}{(a+bx)} = \frac{1}{b^2}\left[a+bx - a\, \ln(a+bx)\right]$$

with $x = t$, $a = b = 1$

$$I_{15} = \left[1 + t - \ln(1+t)\right]\Big|_{\cos \nu_2}^{\cos \nu_1}$$

$$I_{15} = \cos \nu_1 - \cos \nu_2 - \ln\left[\frac{1+\cos \nu_1}{1+\cos \nu_2}\right]$$

A-36:

$$I_{15} = \cos \nu_1 - \cos \nu_2 - [I_{05}]$$

For n = 2: $g_n(t) = 2t^2 - 1$

$$I_{25} = 2\int_{\cos \nu_2}^{\cos \nu_1} \frac{t^2\, dt}{[1+t]} - \int_{\cos \nu_2}^{\cos \nu_1} \frac{dt}{[1+t]}$$

By 36, Ref. (3):

$$\int \frac{x^2\, dx}{a+bx} = \frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2 - 2a\,(a+bx) + a^2 \ln(a+bx)\right]$$

with $x = t$, $a = b = 1$, the first integral of I_{25} $\Rightarrow$

$$2\left[\frac{(1+t)^2}{2} - 2(1+t) + \ln(1+t)\right]\Bigg|_{\cos \nu_2}^{\cos \nu_1}$$

combining with $-I_{05}$

$$I_{25} = 2\left\{\frac{t^2}{2} - t - \frac{3}{2} + \frac{1}{2}\ln(1+t)\right\}\Bigg|_{\cos\nu_2}^{\cos\nu_1}$$

$$I_{25} = \left(\cos^2\nu_1 - \cos^2\nu_2\right) - 2\left(\cos\nu_1 - \cos\nu_2\right) + \ln\left[\frac{1+\cos\nu_1}{1+\cos\nu_2}\right]$$

A-37:

$$I_{25} = \cos^2\nu_1 - \cos^2\nu_2 - 2[I_{15}] - [I_{05}]$$

For n = 3: $g_n(t) = 4t^3 - 3t$

$$I_{35} = 4\int_{\cos\nu_2}^{\cos\nu_1} \frac{t^3\,dt}{[1+t]} - 3\int_{\cos\nu_2}^{\cos\nu_1} \frac{t\,dt}{[1+t]}$$

By the recurrence relation, Equation 2-1112, Ref. (4):

$$\int \frac{x^n\,dx}{(a+bx)} = \frac{x^n}{n_b} - \frac{a}{b}\,\frac{x^{n-1}\,dx}{(a+bx)}$$

with x = t, n = 3, a = b = 1 the first integral of I_{53} yields:

$$\frac{t^3}{3} - \int \frac{t^2\,dx}{(1+t)}$$

From I_{25} and I_{15}:

$$I_{35} = 4\left\{\frac{t^3}{3} - \left[\frac{(1+t)^2}{2} - 2(1+t) + \ln(1+t)\right]\right\}\Bigg|_{\cos\nu_2}^{\cos\nu_1}$$

$$- 3\left[1 + t - \ln(1+t)\right]_{\cos\nu_2}^{\cos\nu_2}$$

combining:

$$I_{35} = \frac{4}{3}\left[(\cos\nu_1)^3 - (\cos\nu_2)^3\right] - 2\left[\cos^2\nu_1 - \cos^2\nu_2\right]$$

$$+ (\cos\nu_1) - (\cos\nu_2) - \ln\left[\frac{1+\cos\nu_1}{1+\cos\nu_2}\right]$$

<u>A-38:</u>

$$I_{35} = \frac{4}{3}\left[(\cos\nu_2)^3 - 2\left[I_{25}\right] - 3\left[I_{15}\right] - 4\left[I_{05}\right]\right.$$

Considering I_{n6}:

$$I_{n6} \equiv \int_{\nu_1}^{\nu_2} \frac{\sin(n\nu)\sin\nu\, d\nu}{[1+\cos\nu]} = \int_{\cos\nu_2}^{\cos\nu_1} \frac{f_n(t)\, dt}{[1+t]}$$

<u>For $n = 0$:</u> $f_n(t) = 0$

<u>A-39:</u>

$$I_{06} = 0$$

For $n = 1$: $f_n(t) = \sqrt{1-t^2}$

$$I_{16} = \int_{\cos\nu_2}^{\cos\nu_1} \frac{\sqrt{1-t^2}}{[1+t]}\, dt = \int_{\cos\nu_2}^{\cos\nu_1} \sqrt{\frac{1-t}{1+t}}\, dt$$

By 60, Ref. (3):

$$\int \frac{\sqrt{v}}{\sqrt{u}}\, dx = \frac{1}{b}\sqrt{uv} - \frac{k}{2b}\left[\frac{2}{\sqrt{-bb'}}\tan^{-1}\sqrt{\frac{-b'u}{bv}}\right.$$

with $v = 1-t = a'+b't$; $a' = 1$, $b' = -1$

$u = 1+t = a+bt$; $a = 1$, $b = 1$

$k = ab' - a'b = -1-1 = -2$

$$\int_{\cos \nu_2}^{\cos \nu_1} \frac{\sqrt{1-t}}{\sqrt{1+t}} = \left[\sqrt{1-t^2} + 2 \tan^{-1} \sqrt{\frac{1+t}{1-t}}\right]_{\cos \nu_2}^{\cos \nu_1}$$

By the identity

$$\cos \frac{1}{2} x = \sqrt{\frac{1+\cos x}{1-\cos x}} = \tan \frac{1}{2} (\pi - x)$$

$$\tan^{-1} \sqrt{\frac{1+\cos \nu_p}{1-\cos \nu_p}} = \frac{1}{2}\left(\pi - \nu_p\right)$$

<u>A-40:</u>

$$I_{16} = \sin \nu_1 - \sin \nu_2 - \nu_1 + \nu_2$$

<u>For n = 2:</u> $f_n(t) = 2t\sqrt{1-t^2}$

$$I_{26} = 2 \int_{\cos \nu_2}^{\cos \nu_1} t \sqrt{\frac{1-t}{1+t}} \, dt = 2 \int_{\cos \nu_2}^{\cos \nu_1} \frac{t(1-t)}{\sqrt{1-t^2}} \, dt$$

by 161, Ref. (3)

$$\int \frac{x dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2}$$

by 171, Ref. (3):

$$\int \frac{x^2 dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

with x = t, a = 1

$$I_{26} = 2 \left\{ \left(-1 + \frac{t}{2}\right)\sqrt{1-t^2} - \frac{1}{2} \sin^{-1} t \right\}_{\cos \nu_2}^{\cos \nu_1}$$

$$I_{26} = 2(\sin \nu_1 - \sin \nu_2) + \cos \nu_1 \sin \nu_1 - \cos \nu_2 \sin \nu_2 -$$

$$-\left(\frac{\pi}{2} \pm \nu_1\right) + \left(\frac{\pi}{2} \pm \nu_2\right)$$

A-41:

$$I_{26} = 2 \sin \nu_2 - 2 \sin \nu_1 + \frac{1}{2} (\sin 2 \nu_1 - \sin 2 \nu_2) - (\pm V_1) + (\pm V_2)$$

Note: Sign is negative for ν_p in first or second quadrants and positive for third and fourth quadrants.

For n = 3: $f_n(t) = 4t^2 \sqrt{1 - t^2} - \sqrt{1 - t^2}$

Following the method used in I_{26}:

$$I_{36} = 4 \int_{\cos \nu_2}^{\cos \nu_1} \frac{t^2(1 - t)\, dt}{\sqrt{1 - t^2}} - \int_{\cos \nu_2}^{\cos \nu_1} \frac{\sqrt{1 - t}}{\sqrt{1+t}}$$

$$= -4 \int_{\cos \nu_2}^{\cos \nu_1} \frac{t^3 dt}{\sqrt{1 - t^2}} + 4 \int_{\cos \nu_2}^{\cos \nu_1} \frac{t^2 dt}{\sqrt{1 - t^2}} - \int_{\cos \nu_2}^{\cos \nu_1} \frac{\sqrt{1 - t}}{\sqrt{1+t}}$$

By 175, Ref. (3):

$$\int \frac{x^3\, dx}{\sqrt{a^2 - x^2}} = \frac{1}{3} \sqrt{(a^2 - x^2)^3} - a^2 \sqrt{a^2 - x^2}$$

with a = 1, x = t, the first integral yields:

$$-\frac{4}{3} (\sin^3 \nu_1 - \sin^3 \nu_2) + 4 (\sin \nu_1 - \sin \nu_2)$$

The second integral, from I_{26} yields

$$-2 \left[I_{26} \right] - 4 (\sin \nu_1 - \sin \nu_2)$$

The third integral $= -\left[I_{16}\right]$

A-42:

$$I_{36} = \frac{4}{3}\left\{\sin^3 \nu_2 - \sin^3 \nu_1\right\} - 2\left[I_{26}\right] - \left[I_{16}\right]$$

Considering I_{n1} and I_{n4}:

These integrals must be solved by use of Elliptic integrals and Jacobian Elliptic Functions.

The solutions for these integrals can be found by transforming them into algebraic form as was done for the integrals I_{n2}, I_{n3}, I_{n5}, and I_{n6}. Upon transformation it is found that they all are of the form solved in series 235 of Ref. (5). In utilizing these solutions, the parameters will have the following values:

$$a = 1,\ b = -1,\ c = -\ell,\ y_p = \cos \nu_p$$

$$k^2 = \frac{a-b}{a-c} = \frac{2}{\ell+1};\quad k'^2 = 1 - k^2 = \frac{\ell-1}{\ell+1}$$

$$g = \frac{2}{\sqrt{a-c}} = \frac{2}{\sqrt{\ell+1}};\quad \frac{2}{k^2} = \sqrt{\ell+1};\quad 1 + k'^2 = \frac{2\ell}{\ell+1}$$

$$\varphi_p = \text{am } u_p = \sin^{-1}\sqrt{\frac{(a-c)(y_p-b)}{(a-b)(y_p-c)}} = \sin^{-1}\sqrt{\frac{(\ell+1)(1+\cos \nu_p)}{2(\ell+\cos \nu_p)}}$$

$u_p = F(\varphi_p, k) =$ Incomplete elliptic integral of the first kind.

$E(\varphi_p, k) = E(u_p) =$ Incomplete elliptic integral of the second kind.

$\pi(\varphi_p, \alpha^2, k) = \pi(u_p, \alpha^2) =$ Incomplete elliptic integral of the third kind.

$$\nu_1 = \pi - (\Phi + \Theta);\quad \nu_2 = \pi - (\varphi - \Theta)$$

Note: Due to the restriction on this group of integrals that

$(a \geq y > b > c)$ $y_p > b = -1$

$\therefore \nu_1, \nu_2 \neq \pi$

Considering I_{n1}:

$$I_{n1} = \int_{\nu_1}^{\nu_2} \frac{\cos n\nu \cos \nu \, d\nu}{[\ell + \cos \nu]^{3/2}}$$

$$I_{n1} = \int_{\cos \nu_2}^{\cos \nu_1} \frac{g_n(t)\, t\, dt}{(\ell+t)\sqrt{1-t^2}\sqrt{\ell+t}}$$

$$I_{n1} = \int_{\cos \nu_2}^{\cos \nu_1} \frac{g_n(t)\, t\, dt}{(\ell+t)\sqrt{1-t}\sqrt{1+t}\sqrt{\ell+t}}$$

For $n = 0$: $g_n(t) = 1$

$$I_{01} = \int_{\cos \nu_2}^{\cos \nu_2} \left[\frac{t}{(\ell+t)}\right] \frac{dt}{\sqrt{1-t}\sqrt{1+t}\sqrt{\ell+t}}$$

Considering the bracketed term:

$$t = (\ell + t) - \ell; \quad \frac{t}{[\ell+t]} = 1 - \frac{\ell}{[\ell+t]}$$

$$I_{01} = \int_{\cos \nu_2}^{\cos \nu_1} \frac{dt}{\sqrt{1-t}\sqrt{t-(-1)}\sqrt{t-(-\ell)}}$$

$$-\ell \int_{\cos \nu_2}^{\cos \nu_1} \frac{dt}{[t-(-\ell)]\sqrt{1-t}\sqrt{t-(-1)}\sqrt{t-(-\ell)}}$$

By 235.00:

$$\int_b^{u_p} \frac{dt}{\sqrt{(a-g)(t-b)(t-c)}} = gF(\varphi_p, k)$$

with $a = 1$, $b = -1$, $c = -\ell$, the first integral in $I_{01} \Rightarrow$

$$\frac{2}{\sqrt{\ell+1}}\left[F(\varphi_1,k)-F(\varphi_2,k)\right]=\frac{2}{\sqrt{\ell+1}}\left[u_1-u_2\right]$$

By 235. 01:

$$\int_0^{y_p}\frac{dt}{(t-c)\sqrt{(a-t)(t-b)(t-c)}}=\frac{g}{(b-c)}E(\varphi_p,k)$$

with $a = 1$, $b = -1$, $c = -\ell$ the second integral in $I_{01} \Rightarrow$

$$\frac{-2\ell}{(\ell-1)\sqrt{\ell+1}}\left[E(\varphi_1,k)-E(\varphi_2,k)\right]$$

A-43:

$$I_{01}=\frac{2}{\sqrt{\ell+1}}\left\{u_1-u_2-\frac{\ell}{\ell-1}\left[E(u_1)-E(u_2)\right]\right\}$$

For n = 1: $g_n(t) = 1$

$$I_{11}=\int_{\cos\nu_2}^{\cos\nu_1}\left[\frac{t^2}{(\ell+t)}\right]\frac{dt}{\sqrt{1-t}\sqrt{1+t}\sqrt{\ell+t}}$$

Considering the bracketed term:

$$\frac{t^2}{(\ell+t)}=\frac{[(\ell+t)-\ell]^2}{(\ell+t)}=\frac{(\ell+t)^2-2\ell(\ell+t)+\ell^2}{(\ell+t)}$$

$$\frac{t^2}{(\ell+t)}=\ell+t-2\ell+\frac{\ell^2}{\ell+t}=t-\ell+\frac{\ell^2}{\ell+t}$$

$$=t-\ell\left[1-\frac{\ell}{(\ell+t)}\right]$$

comparing with I_{01}:

$$I_{11}=\int_{\cos\nu_2}^{\cos\nu_1}\frac{t\,dt}{\sqrt{1-t}\sqrt{t-(-1)}\sqrt{t-(-\ell)}}-\ell\left[I_{01}\right]$$

By 235.16:

$$\int_b^{y_p} \frac{t^m\,dt}{\sqrt{(a-t)(t-b)(t-c)}} = gb^m \int_o^{u_p} \frac{\left(1-\frac{ck^2}{b}\,sn^2\,u\right)^m}{(1-k^2\,sn^2 u)_m}\,du$$

with $m = 1,\ a = 1,\ b = -1,\ c = -\ell$

$$gb^m = \frac{2(-1)}{\sqrt{\ell+1}} = \frac{-2}{\sqrt{\ell+1}}$$

The integral in I_{11} becomes

$$\frac{-2}{\sqrt{\ell+1}} \int_o^{u_1} \frac{(1-\ell k^2 sn^2 u)}{(1-k^2\,sn^2\,u)}\,du\ \frac{+2}{\sqrt{\ell+1}} \int_o^{u_2} \frac{(1-\ell k^2\,sn^2\,u)}{(1-k^2\,sn^2\,u)}\,du$$

By 340.01 with $\alpha^2 = k^2,\ \alpha_1^2 = \ell k^2$

$$\int \frac{1-\ell k^2\,sn^2 u}{1-k^2\,sn^2 u}\,dn = \frac{1}{k^2}\left[(k^2-\ell k^2)\,\pi(\varphi,\,k^2,\,k) + \ell k^2 u\right]$$

$$= \left[(1-\ell)\,\pi\,(\varphi,\,k^2,\,k) + \ell\,u\right]$$

A-44:

$$I_{11} = \frac{2(\ell-1)}{\sqrt{\ell+1}}\left[\pi(\Phi_1,k^2,k) - \pi(\Phi_2,\,k^2,\,k)\right]\ \frac{-2\ell}{\sqrt{\ell+1}}(u_1-u_2) - \ell\left[I_{01}\right]$$

Where, by 111.06, Ref. (5):

$$\pi(\varphi,\,k^2,\,k) = \frac{\left[E(\varphi,k) - (k^2 \sin\varphi\,\cos\varphi/\sqrt{1-k^2\sin^2\varphi}\right]}{k'^2}$$

For n = 2: $g_n = 2t^2 - 1$

$$I_{21} = \int_{\cos\,\nu_2}^{\cos\,\nu_2} \left[\frac{2t^3-t}{(\ell+t)}\right] \frac{dt}{\sqrt{(1-t)\,[t-(-1)]\,[t-(-\ell)]}}$$

Considering the bracketed term:

$$\frac{2t^3 - t}{[\ell+t]} = \frac{2[(\ell+t) - \ell]^3 - [(\ell+t) - \ell]}{\ell + t}$$

$$= \frac{2(\ell+t)^3 - 6(\ell+t)^2\ell + 6(\ell+t)\ell^2 - 2\ell^3 - (\ell+t) + \ell}{(\ell + t)}$$

$$= 2(\ell+t)^2 - 6\ell(\ell+t) + 6\ell^2 - 1 - \left[\frac{2\ell^3 - \ell}{(\ell+t)}\right]$$

$$= 2\ell^2 + 4\ell t + 2t^2 - 6\ell^2 - 6\ell t + 6\ell^2 - 1 - \left[\frac{2\ell^3 - \ell}{(\ell+t)}\right]$$

$$= 2\ell^2 - 2\ell t + 2t^2 - 1 - \left[\frac{2\ell^3 - \ell}{(\ell+t)}\right]$$

$$= 2t^2 - 2\ell\left(t - \ell + \frac{\ell^2}{(\ell+t)}\right) + \frac{\ell}{(\ell+t)} - 1$$

Comparing with I_{11} and I_{01}:

$$I_{21} = 2\int \frac{t^2\, dt}{\sqrt{(1-t)\,[t-(-1)]\,[t-(-\ell)]}} - 2\ell\left[I_{11}\right] - \left[I_{01}\right]$$

Again citing 235.16, with m = 2

$$2\int \frac{t^2\, dt}{\sqrt{(-t)\,[t-(-1)]\,[t-(-\ell)]}} = 2\int_0^{u_1} \frac{(1 - \ell k^2\, sn^2 u)^2\, du}{(1 - k^2\, sn^2\, u)^2}$$

$$-2\int_0^{u_1} \frac{(1 - \ell k^2\, sn^2\, u)^2\, du}{(1 - k^2\, sn^2\, u)^2}$$

By 340.02: with $\alpha^2 = k^2$, $\alpha_1^2 = \ell k^2$

$$\int \frac{(1-\ell k^2 \,sn^2 u)^2 du}{(1-k^2\, sn^2\, u)^2} = \frac{1}{k^4}\left[\ell^2 k^4 u + 2\ell k^2 (k^2 - \ell k^2) V_1 + (k^2 - \ell k^2)^2 V_2\right]$$

By 315: (since $\alpha^2 = k^2$, $V_j = I_{2j}$)

$$V_1(u) = \frac{1}{k'^2}\,[E(u) - k^2 \,sn\, u\, cd\, u]$$

$$V_2(u) = \frac{1}{3k'^4}\,[2(2-k^2)\,E(u) - k'^2 u] - k^2 sn\, u\, cd\, u\,(k'^2\, nd^2\, u + 4 - 2k^2)]$$

$$V_3(u) = \frac{4(2-k^2)\,V_2 - 3V_1 - k^2\, sn\, u\, cd\, u\, nd^4\, u}{5k'^2}$$

Note: V_1, V_2, V_3 are used to evaluate both I_{21}, I_{31}, and I_{34}. They are most readily applied by evaluating them and substituting in their values.

$$\int \frac{(1-\ell k^2 sn^2 u)}{(1-\ell k^2\, sn^2\, u)^2} = \left[\ell^2 u + 2\ell(1-\ell)\,V_1(u) + (1-\ell)^2\, V_2(u)\right]$$

Combining

<u>A-45:</u>

$$I_{21} = 2\ell^2(u_1 - u_2) + 4\ell(1-\ell)\left[V_1(u_1) - V_1(u_2)\right]$$

$$+ 2(1-\ell)^2\left[V_2(u_1) - V_2(u_2)\right] - 2\ell\left[I_{11}\right] - \left[I_{01}\right]$$

<u>For n = 3:</u> $g_n(t) = 4t^3 - 3t$

$$I_{31} = \int_{\cos \nu_2}^{\cos \nu_1} \left[\frac{4t^4 - 3t^2}{(\ell+t)}\right] \frac{dt}{\sqrt{(1-t)\,[t-(-1)]\,[t-(-\ell)]}}$$

Considering the bracketed term:

$$\frac{4t^4 - 3t^2}{(\ell+t)} = \frac{t^2(4t^2 - 3)}{(\ell+t)}$$

$$t^2 = [(\ell + t) - \ell]^2 = (\ell + t)^2 - 2\ell(\ell + t) + \ell^2$$

$$\frac{t^2(4t^2 - 3)}{\ell + t} = t^2 \left[4(\ell+t) - 8\ell + \frac{4\ell^2 - 3}{\ell+t}\right]$$

$$= 4t^3 - 4\ell t^2 + (4\ell^2 - 3)\,[\ell + t) - 2\ell] + \frac{4\ell^4 - 3\ell^2}{(\ell+t)}$$

$$= 4t^3 - 4\ell t^2 + 4\ell^2 t - 3t - 4\ell^3 + 3\ell + \frac{4\ell^4 - 3\ell^2}{(\ell+t)}$$

$$= 4t^3 - 2\ell\left\{2\ell^2 - 2\ell t + 2t^2 - 1 - \left[\frac{2\ell^3 - \ell}{\ell+t}\right]\right\} + \ell - 3t\,\frac{-\ell^2}{(\ell + t)}$$

$$= 4t^3 - 2\ell\left\{2\ell^2 - 2\ell t + 2t^2 - 1 - \frac{2\ell^3 - \ell}{\ell+t}\right]\Big\} + 3\left(\ell - t\,\frac{-\ell^2}{\ell+t}\right) - 2\ell\Big[1 - \frac{\;}{(\ell}$$

comparing with I_{12}, I_{11}, and I_{10}

$$I_{31} = 4\int_{\cos \nu_2}^{\cos \nu_1} \frac{t^3\,dt}{\sqrt{(1 - t)\,[t - (-1)]\,[t - (-\ell)]}} - 2\ell\left[I_{21}\right] - 3\left[I_{11}\right] - 2\ell\left[I_{01}\right]$$

considering the first integral:

By 235. 16, with m = 3:

$$4\int_0^{u_1} \frac{(1 - \ell k^2 \,\mathrm{sn}^2\, u)^3\, du}{(1 - k^2 \,\mathrm{sn}^2\, u)^3} - 4\int_0^{u_2} \frac{(1 - \ell k^2 \,\mathrm{sn}^2\, u)^3\, du}{(1 - k^2 \,\mathrm{sn}^2\, u)^3}$$

By 340.03, with $\alpha^2 = k^2$, $\alpha_1{}^2 = \ell k^2$:

$$\int \frac{(1 - \ell k^2 \operatorname{sn}^2 u)^3}{(1 - k^2 \operatorname{sn}^2 u)^3}\, du = \frac{1}{k^6}\left[\ell^3 k^6 u + 3\ell^2 k^4 (k^2 - \ell k^2) V_1 + 3\ell k^2 (k^2 - \ell k^2)^2 V_2 + (k^2 - \ell k^2)^3 V_3\right]$$

$$\int \frac{(1 - \ell k^2 \operatorname{sn}^2 u)^3}{(1 - k^2 \operatorname{sn}^2 u)^3}\, du = \left[\ell^3 u + 3\ell^2 (1 - \ell) V_1 + 3\ell (1 - \ell)^2 V_2 + (1 - \ell)^3 V_3\right]$$

A-46:

$$\begin{aligned} I_{31} = {} & 4\ell^3 (u_1 - u_2) + 12\ell^2 (1 - \ell)\left[V_1(u_1) - V_1(u_2)\right] \\ & + 12\ell (1 - \ell)^2 \left[V_2(u_1) - V_2(u_2)\right] \\ & + 4(1 - \ell)^3 \left[V_3(u_1) - V_3(u_2)\right] - 2\ell\left[I_{21}\right] - 3\left[I_{11}\right] - 2\ell\left[I_{01}\right] \end{aligned}$$

Considering I_{n4}:

$$I_{n4} = \int_{\nu_1}^{\nu_2} \frac{\sin n\nu \sin \nu \, d\nu}{[1 + \cos \nu]\,[\ell + \cos \nu]^{\frac{1}{2}}}$$

$$= \int_{\cos \nu_2}^{\cos \nu_1} \frac{f_n(t)\, dt}{[1 + t]\,[\ell + t]^{\frac{1}{2}}}$$

Series 235, Ref. (5) will again be used to solve these integrals.

For n = 0: $f_n(t) = 0$

A-47: $I_{04} = 0$

For n = 1: $f_n(t) = \sqrt{(1 - t^2}$

$$I_{14} = \int_{\cos \nu_2}^{\cos \nu_1} \left[\frac{\sqrt{1 - t^2}}{(1 + t)}\right] \frac{dt}{[\ell + t]^{\frac{1}{2}}}$$

considering the bracketed term:

$$\frac{\sqrt{1-t^2}}{1+t} = \frac{\sqrt{1-t}\,\sqrt{1+t}}{1+t} = \frac{\sqrt{1-t}}{\sqrt{1+t}}$$

$$I_{14} = \int_{\cos\nu_2}^{\cos\nu_1} \frac{\sqrt{1-t}\,dt}{\sqrt{(t-(-1)]\,[t-(-\ell)]}}$$

By 235. 07:

$$\int_b^{y_p} \sqrt{\frac{a-t}{(t-b)(t-c)}}\,dt = (a-b)\,g\int_0^{u_p} cd^2 u\,du$$

with a = 1, b = -1, c = $-\ell$

$$I_{14} = \frac{4}{\sqrt{\ell+1}}\int_0^{u_1} cd^2\,u\,du \frac{-4}{\sqrt{\ell+1}}\int_0^{u_2} cd^2\,u\,du$$

By 320. 02:

$$\int cd^2\,u\,du = \frac{1}{k^2}\,[u - E(u) + k^2\,sn\,u\,cd\,u]$$

<u>A-48:</u>

$$I_{14} = 2\sqrt{\ell+1}\left[u_1 - u_2 + E(u_2) - E(u_1) + \frac{2}{\ell+1}(sn\,u_1\,cd\,u_1 - sn\,u_2\,cd\,u_2)\right]$$

<u>For n = 2:</u> $f_n(t) = 2t\sqrt{(1-t^2}$

$$I_{24} = \int_{\cos\nu_2}^{\cos\nu_1}\left[\frac{2t\sqrt{1-t^2}}{(1+t)}\right]\frac{dt}{[\ell+t]^{\frac{1}{2}}}$$

Considering the bracketed term:

$$\frac{2t\sqrt{1-t^2}}{(1+t)} = \frac{2t\sqrt{1-t}}{\sqrt{1+t}} = 2(1+t)\frac{\sqrt{1-t}}{\sqrt{1+t}} - 2\frac{\sqrt{1-t}}{\sqrt{1+t}}$$

$$= 2\sqrt{1+t}\,\sqrt{1-t} - 2\frac{\sqrt{1-t}}{\sqrt{1+t}}$$

Comparing with I_{14}:

$$I_{24} = 2\int_{\cos\nu_2}^{\cos\nu_1} \frac{\sqrt{1+t}\,\sqrt{1-t}}{\sqrt{\ell+t}}\,dt - 2\left[I_{14}\right]$$

By 235.09:

$$\int_b^{y_p} \sqrt{\frac{(a-t)(t-b)}{(t-c)}} = (a-b)(b-c)\,gk^2\int_0^{u_p} sd^2 u\, cd^2 u\, du$$

With $a = 1$, $b = -1$, $c = -\ell$, the integral in $I_{24} \Rightarrow$

$$2\left[2(\ell-1)\left(\frac{2}{\sqrt{\ell+1}}\right)\left(\frac{2}{\ell+1}\right)\right]\left[\int_0^{u_1} sd^2 u\, cd^2 u\, du - \int_0^{u_2} sd^2 u\, cd^2 u\, du\right]$$

By 361.27:

$$\int sd^2 u\, cd^2 u\, du = \frac{1}{3k^4k'^2}\Big[-2k'^2u + (1+k'^2)\,E(u)$$

$$+ k^2(k'^2\,nd^2 u - 1 - k'^2)\,sn\, u\, cd\, u\Big]$$

since:

$$\frac{1}{k^4} = \frac{(\ell+1)^2}{4}, \quad \frac{1+k'^2}{k'^2} = \frac{2\ell}{\ell-1}, \quad \frac{k^2}{k'^2} = \frac{2}{(\ell-1)}$$

$$\int \mathrm{sd}^2\, u\, \mathrm{cd}^2\, u\, du = \frac{(\ell+1)^2}{12}\left[-2u + \frac{2\ell}{\ell-1}\, E(u) + \frac{2}{\ell+1}\right.$$

$$\left.\left(\mathrm{nd}^2\, u - \frac{2\ell}{\ell-1}\right)\mathrm{sn}\, u\, \mathrm{cd}\, u\right]$$

A-49:

$$I_{24} = \frac{4}{3}(\ell-1)\sqrt{\ell+1}\,\left\{2u_2 - 2u_1 + \frac{2\ell}{\ell-1}\left[E(u_1) - E(u_2)\right]\right.$$

$$+\frac{2}{\ell+1}\left[\left(\mathrm{nd}^2\, u_1 - \frac{2\ell}{\ell-1}\right)\mathrm{sn}\, u_1\, \mathrm{cd}\, u_1\right.$$

$$\left.\left.-\left(\mathrm{nd}^2\, u_2 - \frac{2\ell}{\ell-1}\right)\mathrm{sn}\, u_2\, \mathrm{cd}\, u_2\right]\right\} - 2\left[I_{14}\right]$$

For n = 3: $f_n(t) = (4t^2 - 1)\sqrt{1-t^2}$

$$I_{34} = \int_{\cos \nu_2}^{\cos \nu_1} \left[\frac{(4t^2-1)\sqrt{1-t^2}}{(1+t)}\right]\frac{dt}{(\ell+t)^{\frac{1}{2}}}$$

Considering the bracketed term:

$$\frac{(4t^2-1)\sqrt{1-t^2}}{(1+t)} = 2t\,\frac{(2t\sqrt{1+t^2})}{(1+t)} - \frac{\sqrt{1-t^2}}{(1+t)}$$

Substituting $2t = 2(1+t) - 2 \Rightarrow$

$$4t\sqrt{1-t^2} - \frac{4t\sqrt{1-t^2}}{(1+t)} - \frac{\sqrt{1-t^2}}{(1+t)}$$

Comparing with I_{41} and I_{42}:

$$I_{34} = 4\int_{\cos \nu_2}^{\cos \nu_1} \frac{t\sqrt{1-t}\,\sqrt{1+t}}{\sqrt{\ell+t}}\, dt - 2\left[I_{24}\right] - \left[I_{14}\right]$$

with $t = (\ell+t) - \ell$, the integral in I_{43} becomes

$$4\int_{\cos\nu_2}^{\cos\nu_1}\sqrt{(1-t)\,[t-(-1)]\,[t-(-\ell)]}\,dt - 4\ell\int_{\cos\nu_2}^{\cos\nu_1}\frac{\sqrt{(1-t)\,[t-(-1)]}}{\sqrt{t-(-\ell)}}\,dt$$

The second integral is the same one as was solved in I_{42}

$$-4\ell\int_{\cos\nu_2}^{\cos\nu_1}\frac{\sqrt{(1-t)\,[t-(-1)]}}{\sqrt{t-(-\ell)}} = -4\ell\left[\frac{I_{24}+2I_{14}}{2}\right]$$

$$= -2\ell\left[I_{24}\right] - 4\ell\left[I_{14}\right]$$

Considering the first integral:

By 235. 14:

$$\int_0^{y_p}\sqrt{(a-t)(t-b)(t-c)}\,dt = g(a-b)(b-c)^2k^2\int_0^{u_p} sd^2u\,cd^2u\,nd^2u\,du$$

with: $a = 1$, $b = -1$, $c = -\ell$, the first integral $\Rightarrow$

$$4\left(\frac{2}{\sqrt{\ell+1}}\right)(2)\,(\ell-1)^2\left(\frac{2}{\ell+1}\right)\left[\int_0^{u_1} sd^2u\,cd^2u\,nd^2u\,du - \int_0^{u_2} sd^2u\,cd^2u\,nd^2u\,du\right]$$

By 361. 18:

$$\int sd^2u\,cd^2u\,nd^2u\,du = \frac{1}{k^4}\left[-I_2+(1+k'^2)\,I_4 - k'^2\,I_6\right]$$

where $I_2 = V_1$

$I_4 = V_2$ as defined in I_{21}

$I_6 = V_3$

<u>A-50:</u>

$$I_{34} = 8(\ell-1)^2\sqrt{\ell+1}\left\{V_1(u_2) - V_1(u_1) + \frac{2\ell}{\ell+1}\left[V_2(u_1) - V_2(u_2)\right] + \left(\frac{\ell-1}{\ell+1}\right)\right.$$

$$\left.\left[V_3(u_2) - V_3(u_1)\right]\right\} - 2(\ell+1)\left[I_{24}\right] - (4\ell+1)\left[I_{14}\right]$$

APPENDIX C

COMPARISON OF RESULTS WITH KNOWN RESULT FOR A COMPLETE SHROUD

By the Kutta Joukowski Law:

(Equation P-2) $\vec{F}_r(\theta) = -\rho V_{xp}(1, \theta, 0)\,\Gamma_{(\theta)}$

Taking the total lift in the $\theta = 0$ direction:

$$T(0) = -\rho \int_{-\pi}^{\pi} V_{xp}(1, \theta, 0)\,\Gamma_{(\theta)} \cos\theta \; r_{1/4} \, d\theta$$

In the absence of a propeller

$$\vec{V}_{xp} = \vec{V}_\infty$$

$$\therefore T(0) = -\rho V_\infty r_{1/4} \int_{-\pi}^{\pi} \Gamma_{(\theta)} \cos\theta \, d\theta$$

By A-23:

$$\Gamma_{(\theta)} = \sum_{n=o}^{\infty} A_n \cos(n\theta)$$

$$T(0) = -\rho V_\infty r_{1/4} \sum_{n=o}^{\infty} \int_{-\pi}^{\pi} \cos n\theta \cos\theta \, d\theta$$

Because of orthogonality:

$$\int_{-\pi}^{\pi} \cos n\theta \cos\theta \, d\theta = 0, n \neq 1$$

$$\therefore T(0) = -\rho V_\infty r_{1/4} A_1 \int_{-\pi}^{\pi} \cos^2\theta \, d\theta$$

A-51:

$$T(0) = -\rho V_{\infty} r_{1/4} A_1 \pi$$

To find the side thrust on a complete ring, in the absence of a propeller, we need to evaluate A1.

For a complete shroud, of constant chord, at an angle of attack, the side thrust is independent of the section shape (Ref. 6). We will consider the case of a complete shroud consisting of a frustrum of a cone as shown in Fig. A4.

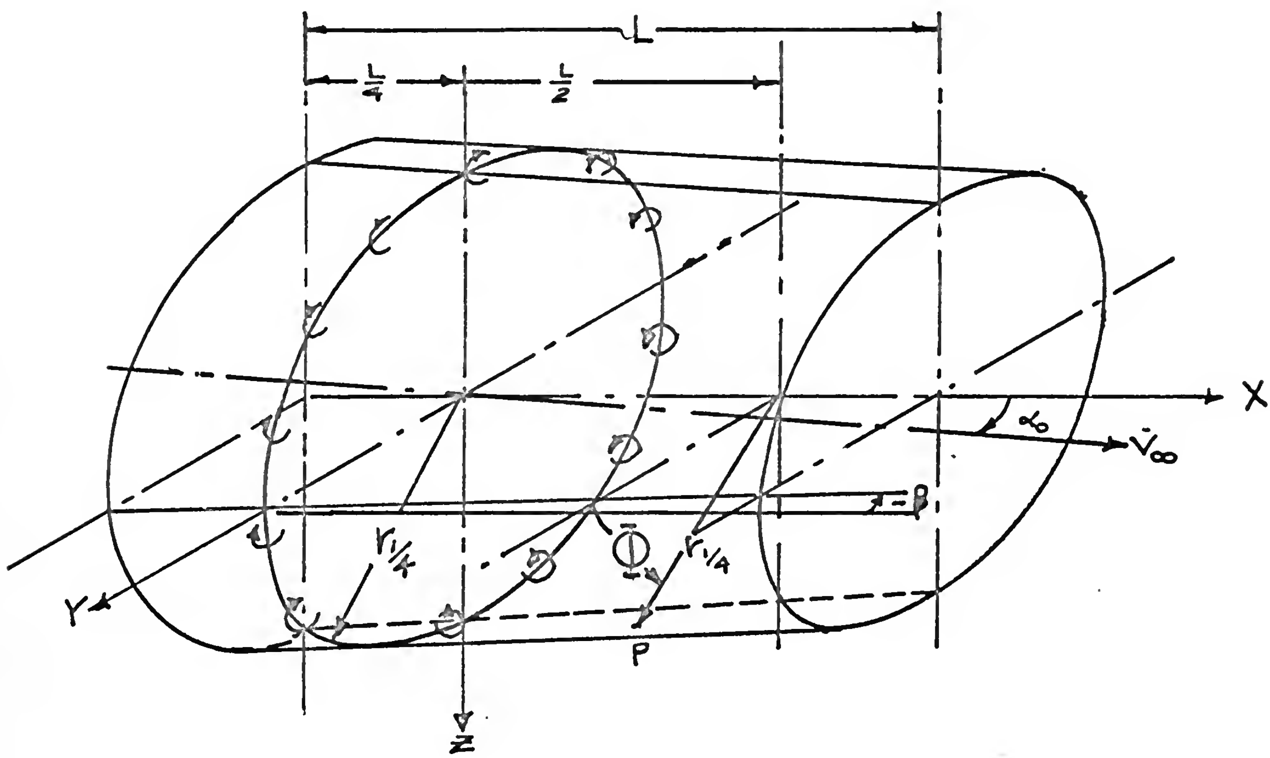


FIGURE A4

The shroud has a chord of L, a radius at the 1/4 chord point of $r_{1/4}$, a convergence angle β and is set at an angle of attack, in the $\theta = 0$ plane of $\alpha(0) = \alpha_0$.

$$\bar{C}'(\Phi, \lambda) = 0$$

$$\alpha(\Phi) = -\beta + \alpha_0 \cos \Phi$$

By P_1:

A-52:

$$\vec{q}_{r_s}\Big|_{(1,\varphi,\lambda_{3/4})} = \vec{r}\, V_\infty \,[+\beta - \alpha_o \cos \Phi]$$

By A-26:

$$\vec{q}_{r_s} = \vec{r} \sum_{n=o}^{\infty} \sum_{m=1}^{6} C_{nm}\, I_{nm}$$

In evaluating the integrals I_{nm} we must abide by the restriction that

$$\nu_1,\ \nu_2 \neq \pi$$

where

$$\nu_1 = \pi - (\Phi + \Theta)$$

$$\nu_2 = \pi - (\Phi - \Theta)$$

Since $\Theta = \pi$, for a complete ring,

$$\Phi \neq \pi$$

$$\nu_1 = -\Phi$$

$$\nu_2 = 2\pi - \Phi \ ; \ \nu_2 - \nu_1 = 2\pi$$

Note: In evaluating the elliptic functions $E(\varphi_p, k)$ and $F(\varphi_p, k)$, the following relations are used:

$$E(-\varphi, k) = -E(\varphi, k)$$

$$F(-\varphi, k) = -F(\varphi, k)$$

$$E(m\pi \pm \varphi, k) = 2m\, E(k) \pm E(\varphi, k)$$

$$F(m\pi \pm \varphi, k) = 2m\, K(k) \pm F(\varphi, k)$$

where:

$E(k)$ = Complete elliptic integral of the second kind.

$F(k)$ = Complete elliptic integral of the first kind.

Determination of φ_1 and φ_2 from ν_1 and ν_2:

$$\varphi_P = \sin^{-1}\sqrt{\frac{(\ell+1)(1+\cos\nu_P)}{2(\ell+\cos\nu_P)}}$$

$$2\sin^2\Phi_P(\ell+\cos\nu_P) = (\ell+1)(1+\cos\nu_P)$$

$$= \ell+\ell\cos\nu_P+1+\cos\nu_P$$

$$(\ell+\cos\nu_P)(2\sin^2\varphi_P-1) = 1+\ell\cos\nu_P$$

$$-\cos(2\varphi_P) = \frac{1+\ell\cos\nu_P}{\ell+\cos\nu_P}$$

when ν_P changes by 2π, $|2\varphi_P|$ changes by 2π. To find the sign, assume $\ell = 2$.

(1) let $\nu_{P_1} = 0$

$$-\cos 2\varphi_{P_1} = \frac{1+2}{2+1} = 1\ ;\ \cos 2\varphi_{P_1} = -1\ ;\ \varphi_{P_1} = \frac{\pi}{2}$$

(2) let $\nu_{P_2} = \pi/3$

$$-\cos 2\varphi_P = \frac{1+1}{2+\frac{1}{2}} = 0.8$$

$$2\varphi_{P_2} = \pi - 37° \quad \varphi_{P_2} \quad \varphi_{P_2} = \frac{\pi}{2} - 18°$$

(3) let $\nu_{P_3} = \pi/2$

$$-\cos 2\varphi_P = \tfrac{1}{2}\ ;\ \cos 2\varphi_P = -\tfrac{1}{2}$$

$$2\varphi_P = 2\pi/3 \quad \varphi_P = \pi/3$$

(4) let $\nu p_4 = \pi$

$$-\cos 2\varphi p_4 = -1/1 = -1$$

$$\cos 2\varphi p_4 = 1 \quad \varphi p_4 = 0$$

when

νp changes by $+2\pi$

φp changes by $-\pi$ as shown below:

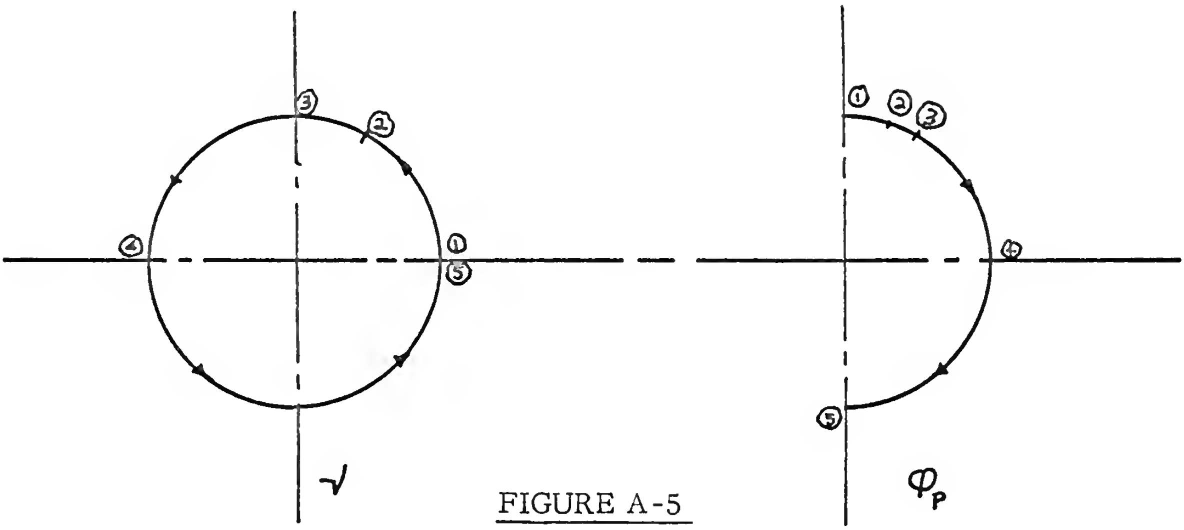


FIGURE A-5

$$E(\varphi p - \pi, k) - E(\varphi p, k) = -2E(k) + E(\varphi p, k) - E(\varphi p, k)$$

$$E(\Phi p - \pi, k) - E(\varphi p, k) = -2E(k)$$

Similarly

$$F(\varphi p - \pi, k) - F(\varphi p, k) = -2K(k)$$

Since in this case

$$\nu_2 - \nu_1 = 2\pi$$

$$\Phi_2 - \varphi_1 = -\pi \qquad \Phi_2 = \Phi_1 - \pi$$

$$E(u_2) - E(u_1) = E(\varphi_1 - \pi, k) - E(\varphi_1, k)$$

$$E(u_2) - E(u_1) = -2E(k)$$

Similarly

$$u_2 - u_1 = -2K(k)$$

By A-43:

$$I_{01} = \frac{2}{\sqrt{\ell + 1}} \left\{ +2\,K(k) - \frac{2\ell}{\ell - 1}\; E(k) \right\}$$

where

$$\ell = \frac{\lambda^2 + 2}{2} \quad , \quad \ell - 1 = \frac{\lambda^2}{2}$$

$$k^2 = \frac{2}{\ell + 1} = \frac{2}{\dfrac{\lambda^2 + 2 + 1}{2}} = \frac{4}{(\lambda^2 + 4)}$$

$$k'^2 = 1 - k^2 = \frac{\ell - 1}{\ell + 1}$$

$$(\ell + 1) = \frac{\lambda^2 + 4}{2} \quad ; \quad \sqrt{\ell + 1} = \frac{\sqrt{\lambda^2 + 4}}{\sqrt{2}}$$

$$\frac{\ell}{\ell - 1} = \frac{\lambda^2 + 2}{\lambda^2}$$

$$I_{01} = \frac{4\sqrt{2}}{\sqrt{\lambda^2 + 4}} \left[-\left(\frac{\lambda^2 + 2}{\lambda^2} \right) E(k) + K(k) \right]$$

$$I_{01} = 4\sqrt{2} \left[-\left(\frac{\lambda^2 + 2}{\lambda^2 \sqrt{\lambda^2 + 4}} \right) E(k) + \frac{K(k)}{\sqrt{\lambda^2 + 4}} \right]$$

<u>A-53:</u>

$$\sum_{m=1}^{6} C_{om}\, I_{om} = C_{01}\, I_{01}$$

$$= \frac{-A_o}{\pi r_{1/4}} \left[-\left(\frac{\lambda^2 + 2}{2\lambda \sqrt{\lambda^2 + 4}} \right) E(k) + \frac{\lambda K(k)}{2\sqrt{\lambda^2 + 4}} \right]$$

For n = 1:

$$I_{12} = I_{13} = I_{15} = 0$$

By A-44:

$$I_{11} = \frac{2(\ell - 1)}{\sqrt{\ell + 1}} \left[\frac{+2E(k)}{k'^2}\right]$$

$$- \frac{2\ell}{\sqrt{\ell+1}} [+2K(k)] - \ell [I_{01}]$$

Since

$$k'^2 = \left(\frac{\ell - 1}{\ell+1}\right)$$

$$I_{11} = \frac{2}{\sqrt{\ell+1}} \left\{+2(\ell+1)\, E(k) + \left(\frac{2\ell^2}{\ell - 1}\right) E(k) - 2\ell\, K(k) - 2\ell K(k)\right\}$$

$$I_{11} = \frac{4}{\sqrt{\ell+1}} \left\{+\left[\left(\frac{\ell^2}{\ell - 1}\right) + (\ell + 1)\right] E(k) - 2\ell\, K(k)\right\}$$

By A-48:

$$I_{14} = 2\sqrt{\ell + 1}\, [+2\, K(k) - 2E(k)]$$

$$I_{14} = \frac{4}{\sqrt{\ell+1}} \, [-(\ell+1)\, E(k) + (\ell+1)\, K(k)]$$

By A-40:

$$I_{16} = +2\pi$$

$$\sum_{m=1}^{6} C_{1m}I_{1m} = C_{11}I_{11} + C_{14}I_{14} + C_{16}I_{16}$$

$$= \frac{\lambda A_1 \cos \varphi}{8\sqrt{2}\pi r_{1/4}} \left[I_{11} + I_{14} + \frac{\sqrt{2}}{\lambda} I_{16} \right]$$

$$I_{11} + I_{14} = \frac{4}{\sqrt{\ell+1}} \left\{ + \left(\frac{\ell^2}{\ell - 1} \right) E(k) - (\ell - 1) K(k) \right\}$$

$$\ell = \frac{\lambda^2 + 2}{2} \; ; \; \ell - 1 = \frac{\lambda^2}{2} \; ; \; \ell + 1 = \frac{\lambda^2 + 4}{2} \; ; \; \frac{\ell^2}{\ell - 1} = \frac{(\lambda^2 + 2)^2}{2\lambda^2}$$

$$I_{11} + I_{14} = \frac{2\sqrt{2}}{\sqrt{\lambda^2 + 4}} \left\{ + \frac{(\lambda^2 + 2)^2}{\lambda^2} \; E(k) - \lambda^2 K(k) \right\}$$

<u>A-54:</u>

$$\sum_{m=1}^{6} C_{1m}I_{1m} = \frac{A_1 \cos \Phi}{\pi r_{1/4}} \left\{ + \frac{(\lambda^2 + 2)^2}{4\lambda\sqrt{\lambda^2 + 4}} \; E(k) \; \frac{-\lambda^3}{4\sqrt{\lambda^2 + 4}} + \frac{\pi}{4} \right\}$$

Similarly, it can be seen from the forms of C_{nm} and I_{nm} that for a complete shroud:

$$\sum_{m=1}^{6} C_{2m} I_{2m} = - \frac{A_2 \cos 2\Phi}{\pi r_{1/4}} [H_2(\lambda, k)]$$

and in general:

<u>A-55:</u>

$$\sum_{m=1}^{6} C_{nm} I_{nm} = \frac{(-1)^{n+1} A_n \cos n\Phi}{\pi r_{1/4}} [H_n(\lambda, k)]$$

where $H_n(\lambda, k)$ is some function of λ and k to be evaluated. Substituting A-53, A-54, and A-55 into A-26, and equating A-52 to A-26:

$$V_\infty [\beta - \alpha_o \cos \Phi] = - \frac{A_o}{\pi r_{1/4}} \left[- \left(\frac{\lambda^2 + 2}{2\lambda \sqrt{\lambda^2 + 4}} \right) E(k) + \frac{\lambda K(k)}{2 \sqrt{\lambda^2 + 4}} \right]$$

$$+ \frac{A_1 \cos \Phi}{\pi r_{1/4}} \left\{ + \left[\frac{(\lambda^2 + 2)^2}{4\lambda \sqrt{\lambda^2 + 4}} \right] E(k) - \frac{\lambda^3 K(k)}{4 \sqrt{\lambda^2 + 4}} + \frac{\pi}{4} \right\}$$

$$+ \sum_{n=2}^{\infty} \frac{(-1)^n A_n \cos n\Phi}{\pi r_{1/4}} [H_n(\lambda, k)]$$

Equating coefficients:

$$A_o = \frac{-V_\infty \pi r_{1/4} \beta}{\left[\left(\frac{\lambda^2 + 2}{2\lambda \sqrt{\lambda^2+4}} \right) E(k) - \frac{\lambda K(k)}{2 \sqrt{\lambda^2 + 4}} \right]}$$

$$A_1 = \frac{-V_\infty \pi r_{1/4} \alpha_o}{\left\{ + \left[\frac{(\lambda^2 + 2)^2}{4\lambda \sqrt{\lambda^2+4}} \right] E(k) - \frac{\lambda^3 K(k)}{4 \sqrt{\lambda^2 + 4}} + \frac{\pi}{4} \right\}}$$

$$A_n = 0, \; n \geq 2$$

Substituting A_1 into A-51:

$$T(0) = \frac{+\rho V_\infty^2 \pi^2 r_{1/4}^2 \alpha_o}{\left\{ + \left[\frac{(\lambda^2 + 2)^2}{4\lambda \sqrt{\lambda^2 + 4}} \right] E(k) - \frac{\lambda^3 K(k)}{4\sqrt{\lambda^2 + 4}} + \frac{\pi}{4} \right\}}$$

Recalling:

$$\lambda = \frac{(L/2)}{r_{1/4}} \quad ; \quad k^2 = \left(\frac{4}{\lambda^2 + 4}\right)$$

and utilizing a lift coefficient analogous to that used in Ref. (2):

$$C_L = \frac{T(0)}{(\rho/2)\, V_\infty^2\, L\, r_{1/4}} = \frac{T(0)}{\rho\, V_\infty^2\, \lambda\, r_{1/4}^2}$$

$$\frac{C_L}{\alpha_o} = \frac{+\pi^2}{\lambda\left\{+\left[\frac{(\lambda^2+2)^2}{4\lambda\sqrt{\lambda^2+4}}\right]E(k) - \frac{\lambda^3 K(k)}{4\sqrt{\lambda^2+4}} + \frac{\pi}{4}\right\}}$$

A-56:

$$\frac{C_L}{\alpha_o} = \frac{+4\pi^2}{\lambda\left\{\left[\frac{\lambda^2+2)^2}{\lambda\sqrt{\lambda^2+4}}\right]E(k) - \frac{\lambda^3 K(k)}{\sqrt{\lambda^2+4}} + \pi\right\}}$$

This result is equivalent to that obtained by Weissinger in Ref. (1) and, as shown in Figure A-6 is very close to the result obtained from the more exact theory by both Weissinger and Morgan in Ref. (2).

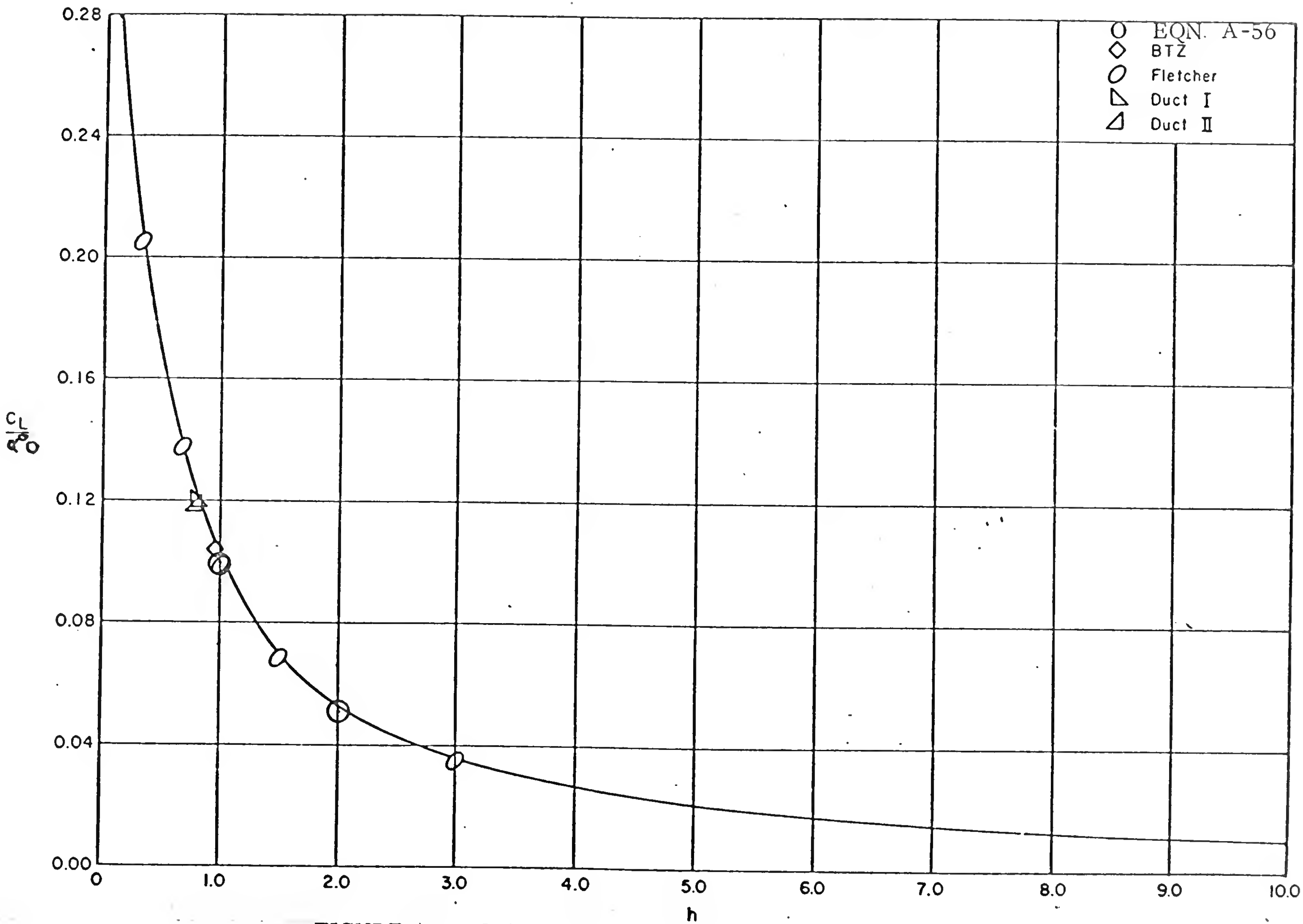


FIGURE A- : Lift Curve Slope as Function of Chord-Diameter Ratio.

Taken from "Prediction of the Aerodynamic Characteristics of Annular Airfoils" by W. B Morgan and E. B Caster.
(DTMB RPT 1830)